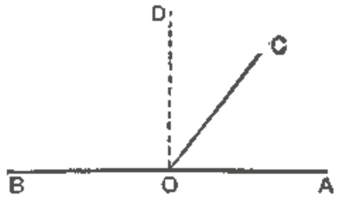
THEOREM 1. [Euclid I. 13]

The adjacent angles which one straight line makes with another straight line on one side of it, are together equal to two right angles.



Let the straight line CO make with the straight line AB the adjacent $\angle^s$ AOC, COB.

It is required to prove that the $\angle^s$ AOC, COB *are together equal to two right angles.*

Suppose OD is at right angles to BA.

Proof. Then the $\angle^s$ AOC, COB together
= the three $\angle^s$ AOC, COD, DOB.
Also the $\angle^s$ AOD, DOB together
= the three $\angle^s$ AOC, COD, DOB.
$\therefore$ the $\angle^s$ AOC, COB together = the $\angle^s$ AOD, DOB
= two right angles.

Q.E.D.

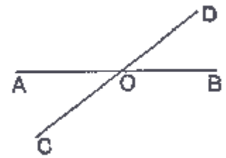
PROOF BY ROTATION

Suppose a straight line revolving about O turns from the position OA into the position OC, and thence into the position OB; that is, let the revolving line turn in succession through the $\angle^s$ AOC, COB.

Now in passing from its first position OA to its final position OB, the revolving line turns through two right angles, for AOB is a straight line.

Hence the $\angle^s$ AOC, COB together = two right angles.

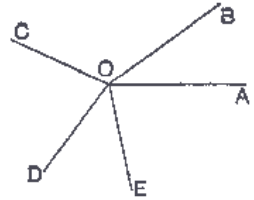
COROLLARY 1. *If two straight lines cut one another, the four angles so formed are together equal to four right angles.*



For example,

$\angle$ BOD + $\angle$ DOA + $\angle$ AOC + $\angle$ COB = 4 right angles.

COROLLARY 2. *When any number of straight lines meet at a point, the sum of the consecutive angles so formed is equal to four right angles.*



For a straight line revolving about O, and turning in succession through the $\angle^s$ AOB, BOC, COD, DOE, EOA, will have made one complete revolution, and therefore turned through four right angles.

DEFINITIONS

(i) Two angles whose sum is *two* right angles, are said to be **supplementary**; and each is called the **supplement** of the other.

Thus in the Fig. of Theor. 1 the angles AOC, COB are supplementary. Again the angle 123° is the supplement of the angle 57°.

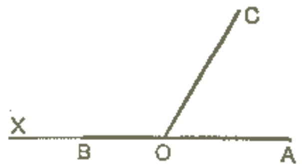
(ii) Two angles whose sum is *one* right angle are said to be **complementary**; and each is called the **complement** of the other.

Thus in the Fig. of Theor. 1 the angle DOC is the complement of the angle AOC. Again angles of 34° and 56° are complementary.

COROLLARY 3. (i) *Supplements of the same angle are equal.*
(ii) *Complements of the same angle are equal.*

Theorem 2. [Euclid I. 14]

If, at a point in a straight line, two other straight lines, on opposite sides of it, make the adjacent angles together equal to two right angles, then these two straight lines are in one and the same straight line.



At O in the straight line CO let the two straight lines OA, OB, on opposite sides of CO, make the adjacent ∠s AOC, COB together equal to two right angles: (that is, let the adjacent ∠s AOC, COB be supplementary).

It is required to prove that OB *and* OA *are in the same straight line.*

Produce AO beyond O to any point X: it will be shewn that OX and OB are the same line.

Proof. Since by construction AOX is a straight line,
∴ the ∠COX is the supplement of the ∠COA. *Theor.* 1.

But, by hypothesis,
the ∠COB is the supplement of the ∠COA.

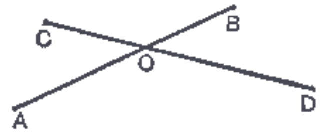
∴ the ∠COX = the ∠COB;
∴ OX and OB are the same line.

But, by construction, OX is in the same straight line with OA;
hence OB is also in the same straight line with OA.

Q.E.D.

Theorem 3. [Euclid I. 15]

If two straight lines cut one another, the vertically opposite angles are equal.



Let the straight lines AB, CD cut one another at the point O.

It is required to prove that

(i) *the* ∠AOC = *the* ∠DOB;

(ii) *the* ∠COB = *the* ∠AOD.

Proof. Because AO meets the straight line CD,
∴ the adjacent ∠s AOC, AOD together = two right angles;
that is, the ∠AOC is the supplement of the ∠AOD.

Again, because DO meets the straight line AB,
∴ the adjacent ∠s DOB, AOD together = two right angles;
that is, the ∠DOB is the supplement of the ∠AOD.

Thus each of the ∠s AOC, DOB is the supplement of the ∠AOD,
∴ the ∠AOC = the ∠DOB.

Similarly, the ∠COB = the ∠AOD.

Q.E.D.

Proof by Rotation

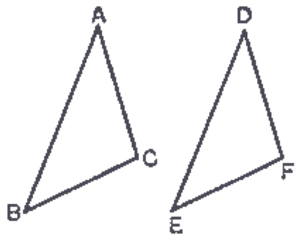
Suppose the line COD to revolve about O until OC turns into the position OA. Then at the same moment OD must reach the position OB (for AOB and COD are *straight*).

Thus the same amount of turning is required to close the ∠AOC as to close the ∠DOB.

∴ the ∠AOC = the ∠DOB.

THEOREM 4. [Euclid I. 4]

If two triangles have two sides of the one equal to two sides of the other, each to each, and the angles included by those sides equal, then the triangles are equal in all respects.



Let ABC, DEF be two triangles in which

AB = DE,
AC = DF,
and the included angle BAC = the included angle EDF.

It is required to prove that the △ABC = *the* △DEF *in all respects.*

Proof. Apply the △ABC to the △DEF,
so that the point A falls on the point D,
and the side AB along the side DE.

Then because AB = DE,
∴ the point B must coincide with the point E.

And because AB falls along DE,
and the ∠BAC = the ∠EDF,
∴ AC must fall along DF.

And because AC = DF,
∴ the point C must coincide with the point F.

Then since B coincides with E, and C with F,
∴ the side BC must coincide with the side EF.

Hence the △ABC coincides with the △DEF,
and is therefore equal to it in all respects.

Q.E.D.

Obs. In this Theorem we must carefully observe what is *given* and what is *proved.*

Given that { AB = DE, AC = DF, and the ∠BAC = the ∠EDF.

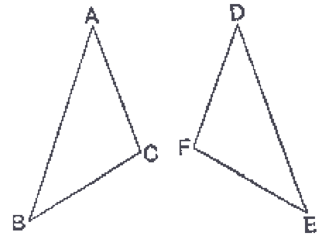
From these data we prove that the triangles coincide on superposition.

Hence we *conclude* that { BC = EF, the ∠ABC = the ∠DEF, and the ∠ACB = the ∠DFE;

also that the triangles are equal in area.

Notice that the angles which are proved equal in the two triangles are opposite to sides which were given equal.

NOTE. The adjoining diagram shews that in order to make two congruent triangles coincide, it may be necessary to *reverse*, that is, turn over one of them before superposition.



EXERCISES

1. *Shew that the bisector of the vertical angle of an isosceles triangle* (i) *bisects the base,* (ii) *is perpendicular to the base.*

2. *Let* O *be the middle point of a straight line* AB, *and let* OC *be perpendicular to it. Then if* P *is any point in* OC, *prove that* PA = PB.

3. Assuming that the four sides of a square are equal, and that its angles are all right angles, prove that in the square ABCD, the diagonals AC, BD are equal.

4. ABCD is a square, and L, M, and N are the middle points of AB, BC, and CD: prove that

(i) LM = MN. (ii) AM = DM.
(iii) AN = AM. (iv) BN = DM.

[Draw a separate figure in each case.]

5. ABC is an isosceles triangle: from the equal sides AB, AC two equal parts AX, AY are cut off, and BY and CX are joined. Prove that BY = CX.

(Numerical)

1. Through what angle does the minute-hand of a clock turn in (i) 5 minutes, (ii) 21 minutes, (iii) $43\frac{1}{2}$ minutes, (iv) 14 min. 10 sec.? And how long will it take to turn through (v) 66°, (vi) 222°?

2. A clock is started at noon: through what angles will the hour-hand have turned by (i) 3.45, (ii) 10 minutes past 5? And what will be the time when it has turned through $172\frac{1}{2}$°?

3. The earth makes a complete revolution about its axis in 24 hours. Through what angle will it turn in 3 hrs. 20 min., and how long will it take to turn through 130°?

4. In the diagram of Theorem 3

(i) If the ∠AOC = 35°, write down (without measurement) the value of each of the ∠s COB, BOD, DOA.

(ii) If the ∠s COB, AOD together make up 250°, find each of the ∠s COA, BOD.

(iii) If the ∠s AOC, COB, BOD together make up 274°, find each of the four angles at O.

(Theoretical)

5. If from O a point in AB two straight lines OC, OD are drawn on opposite sides of AB so as to make the angle COB equal to the angle AOD; shew that OC and OD are in the same straight line.

6. Two straight lines AB, CD cross at O. If OX is the bisector of the angle BOD, prove that XO produced bisects the angle AOC.

7. Two straight lines AB, CD cross at O. If the angle BOD is bisected by OX, and AOC by OY, prove that OX, OY are in the same straight line.

8. If OX bisects an angle AOB, shew that, by folding the diagram about the bisector, OA may be made to coincide with OB.

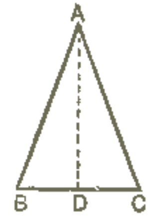
How would OA fall with regard to OB, if
(i) the ∠AOX were greater than the ∠XOB;
(ii) the ∠AOX were less than the ∠XOB?

9. AB and CD are straight lines intersecting at right angles at O; shew by folding the figure about AB, that OC may be made to fall along OD.

10. A straight line AOB is drawn on paper, which is then folded about O, so as to make OA fall along OB; shew that the crease left in the paper is perpendicular to AB.

THEOREM 5. [Euclid I. 5]

The angles at the base of an isosceles triangle are equal.



Let ABC be an isosceles triangle, in which the side AB = the side AC.

It is required to prove that the ∠ABC = *the* ∠ACB.

Suppose that AD is the line which bisects the ∠BAC, and let it meet BC in D.

1st Proof. Then in the △s BAD, CAD,

because { BA = CA,
AD is common to both triangles,
and the included ∠BAD = the included ∠CAD;

∴ the triangles are equal in all respects; *Theor.* 4.
so that the ∠ABD = the ∠ACD.

Q.E.D.

2nd Proof. Suppose the △ABC to be folded about AD.
Then since the ∠BAD = the ∠CAD,
∴ AB must fall along AC.
And since AB = AC,
∴ B must fall on C, and conseqently DB on DC.
∴ the ∠ABD will coincide with the ∠ACD, and is therefore equal to it.

Q.E.D.

EXERCISES

1. Write down the supplements of *one-half* of a right angle, *four-thirds* of a right angle; also of 46°, 149°, 83°, 101° 15′.

2. Write down the complement of *two-fifths* of a right angle; also of 27°, 38° 16′, and 41° 29′ 30″.

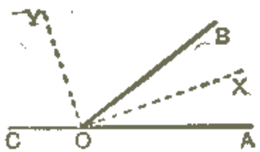
3. If two straight lines intersect forming four angles of which one is known to be a right angle, prove that the other three are also right angles.

4. In the triangle ABC the angles ABC, ACB are given equal. If the side BC is produced both ways, shew that the exterior angles so formed are equal.

5. In the triangle ABC the angles ABC, ACB are given equal. If AB and AC are produced beyond the base, shew that the exterior angles so formed are equal.

Definition. The lines which bisect an angle and the adjacent angle made by producing one of its arms are called the **internal** and **external bisectors** of the given angle.

Thus in the diagram, OX and OY are the internal and external bisectors of the angle AOB.



6. Prove that the bisectors of the adjacent angles which one straight line makes with another contain a right angle. That is to say, *the internal and external bisectors of an angle are at right angles to one another.*

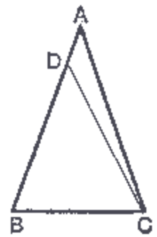
7. Shew that the angles AOX and COY in the above diagram are complementary.

8. Shew that the angles BOX and COX are supplementary; and also that the angles AOY and BOY are supplementary.

9. If the angle AOB is 35°, find the angle COY.

Theorem 6. [Euclid I. 6]

If two angles of a triangle are equal to one another, then the sides which are opposite to the equal angles are equal to one another.



Let ABC be a triangle in which
the ∠ABC = the ∠ACB.

It is required to prove that the side AC = *the side* AB.

If AC and AB are not equal, suppose that AB is the greater.
From BA cut off BD equal to AC.
Join DC.

Proof. Then in the $\triangle^s$ DBC, ACB,

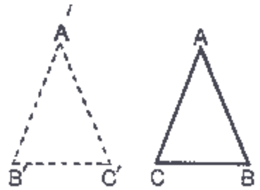
because { DB = AC,
BC is common to both,
and the included ∠DBC = the included ∠ACB;

∴ the △DBC = the △ACB in area, *Theor.* 4.
the part equal to the whole; which is absurd.
∴ AB is not unequal to AC;
that is, AB = AC.

Q.E.D.

Corollary. *Hence if a triangle is equiangular it is also equilateral.*

NOTE ON THEOREMS 5 AND 6



Theorems 5 and 6 may be verified experimentally by cutting out the given △ABC, and, after turning it over, fitting it *thus reversed* into the vacant space left in the paper.

Suppose A′B′C′ to be the original position of the △ABC, and let ACB represent the triangle when reversed.

In Theorem 5, it will be found on applying A to A′ that C may be made to fall on B′, and B on C′.

In Theorem 6, on applying C to B′ and B to C′ we find that A will fall on A′.

In either case the given triangle *reversed* will coincide with its own "trace," so that the side and angle on the *left* are respectively equal to the side and angle on the *right*.

NOTE ON A THEOREM AND ITS CONVERSE

The enunciation of a theorem consists of two clauses. The first clause tells us what we are to *assume*, and is called the **hypothesis**; the second tells us what *it is required to prove*, and is called the **conclusion**.

For example, the enunciation of Theorem 5 assumes that in a certain triangle ABC *the side* AB = *the side* AC: this is the *hypothesis*. From this it is required to prove that *the angle* ABC = *the angle* ACB: this is the *conclusion*.

If we interchange the hypothesis and conclusion of a theorem, we enunciate a new theorem which is called the **converse** of the first.

For example, in Theorem 5

it is *assumed* that AB = AC;
it is *required to prove* that the angle ABC = the angle ACB.

Now in Theorem 6

it is *assumed* that the angle ABC = the angle ACB;
it is *required to prove* that AB = AC.

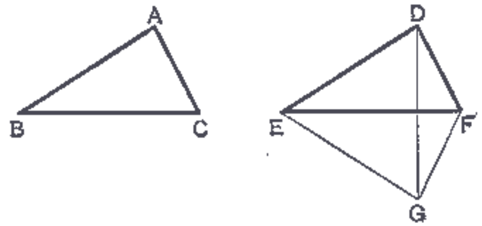
Thus we see that Theorem 6 is the converse of Theorem 5; for the *hypothesis of each is the conclusion of the other*.

In Theorem 6 we employ an *indirect method of proof* frequently used in geometry. It consists in shewing that the theorem *cannot be untrue*; since, if it were we should be led to some *impossible conclusion*. This form of proof is known as **Reductio ad Absurdum**, and is most commonly used in demonstrating the converse of some foregoing theorem.

It must not however be supposed that if a theorem is true, its converse is *necessarily* true. [See p. 25.]

THEOREM 7 [Euclid I. 8]

If two triangles have the three sides of the one equal to the three sides of the other, each to each, they are equal in all respects.



Let ABC, DEF be two triangles in which

AB = DE,
AC = DF,
BC = EF.

It is required to prove that the triangles are equal in all respects.

Proof. Apply the △ABC to the △DEF,
so that B falls on E, and BC along EF, and
so that A is on the side of EF opposite to D.
Then because BC = EF, C must fall on F.

Let GEF be the new position of the △ABC.
Join DG.

Because ED = EG,
∴ the ∠EDG = the ∠EGD. *Theor.* 5.

Again, because FD = FG,
∴ the ∠FDG = the ∠FGD.

Hence the whole ∠EDF = the whole ∠EGF,
that is, the ∠EDF = the ∠BAC.

Then in the △s BAC, EDF;

because { BA = ED,
AC = DF,
and the included ∠BAC = the included ∠EDF;

∴ the triangles are equal in all respects. *Theor.* 4.

Q.E.D.

Obs. In this Theorem
it is *given* that AB = DE, BC = EF, CA = FD;
and we *prove* that ∠C = ∠F, ∠A = ∠D, ∠B = ∠E.
Also the triangles are equal in area.

Notice that the angles which are proved equal in the two triangles are opposite to sides which were given equal.

NOTE 1. We have taken the case in which DG falls within the ∠s EDF, EGF.

Two other cases might arise:

(i) DG might fall outside the ∠s EDF, EGF [as in Fig. 1].

(ii) DG might coincide with DF, FG [as in Fig. 2].

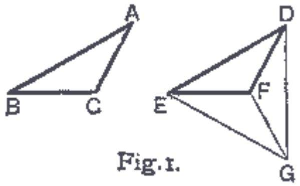


Fig. 1.

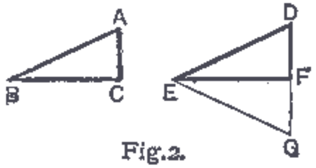


Fig. 2.

These cases will arise only when the given triangles are obtuse-angled or right-angled; and (as will be seen hereafter) not even then, if we begin by choosing for superposition the *greatest* side of the △ABC, as in the diagram of page 24.

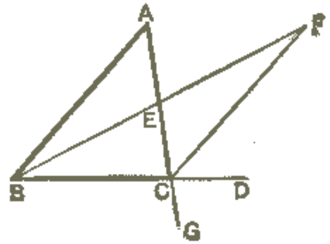
NOTE 2. Two triangles are said to be equiangular to one another when the angles of one are respectively equal to the angles of the other.

Hence if two triangles have the three sides of one severally equal to the three sides of the other, the triangles are equiangular to one another.

The student should state the converse theorem, and shew by a diagram that the converse is not necessarily true.

THEOREM 8. [Euclid I. 16]

If one side of a triangle is produced, then the exterior angle is greater than either of the interior opposite angles.



Let ABC be a triangle, and let BC be produced to D.

It is required to prove that the exterior ∠ACD *is greater than either of the interior opposite* ∠s ABC, BAC.

Suppose E to be the middle point of AC.
Join BE; and produce it to F, making EF equal to BE.
Join FC.

Proof. Then in the △s AEB, CEF,

because { AE = CE,
EB = EF,
and the ∠AEB = the vertically opposite ∠CEF;

∴ the triangles are equal in all respects; *Theor.* 4
so that the ∠BAE = the ∠ECF.

But the ∠ECD is greater than the ∠ECF;
∴ the ∠ECD is greater than the ∠BAE;
that is, the ∠ACD is greater than the ∠BAC.

In the same way, if AC is produced to G, by supposing A to be joined to the middle point of BC, it may be proved that the ∠BCG is greater than the ∠ABC.

But the ∠BCG = the vertically opposite ∠ACD.
∴ the ∠ACD is greater than the ∠ABC.

Q.E.D.

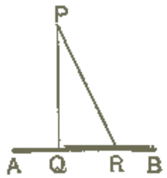
Corollary 1. *Any two angles of a triangle are together less than two right angles.*

For the ∠ABC is less than the ∠ACD : *Proved.*
to each add the ∠ACB.
Then the ∠s ABC, ACB are less than the ∠s ACD, ACB,
therefore, less than two right angles.

Corollary 2. *Every triangle must have at least two acute angles.*

For if one angle is obtuse or a right angle, then by Cor. I each of the other angles must be less than a right angle.

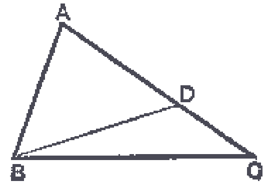
Corollary 3. *Only one perpendicular can be drawn to a straight line from a given point outside it.*



If two perpendiculars could be drawn to AB from P, we should have a triangle PQR in which each of the ∠s PQR, PRQ would be a right angle, which is impossible.

Theorem 9. [Euclid I. 18]

If one side of a triangle is greater than another, then the angle opposite to the greater side is greater than the angle opposite to the less.



Let ABC be a triangle, in which the side AC is greater than the side AB.

It is required to prove that the ∠ABC *is greater than the* ∠ACB.

From AC cut off AD equal to AB.
Join BD.

Proof. Because AB = AD,
∴ the ∠ABD = the ∠ADB. *Theor.* 5.

But the exterior ∠ADB of the △BDC is greater than the interior opposite ∠DCB, that is, greater than the ∠ACB.

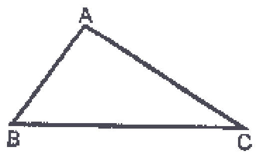
∴ ∠ABD is greater than the ∠ACB
Still more then is ∠ABC greater than the ∠ACB.

Q.E.D.

Obs. The mode of demonstration used in the following Theorem is known as the **Proof by Exhaustion.** It is applicable to cases in which *one* of certain suppositions must necessarily be true; and it consists in shewing that each of these suppositions is false *with one exception*: hence the truth of the remaining supposition is inferred.

THEOREM 10. [Euclid I. 19]

If one angle of a triangle is greater than another, then the side opposite to the greater angle is greater than the side opposite to the less.



Let ABC be a triangle, in which the ∠ABC is greater than the ∠ACB.

It is required to prove that the side AC *is greater than the side* AB.

Proof. If AC is not greater than AB,
it must be either equal to, or less than AB.

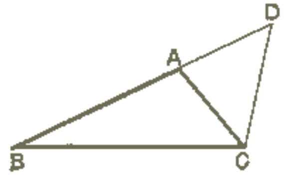
Now if AC were equal to AB,
then the ∠ABC would be equal to the ∠ACB ; *Theor.* 5.
but, by hypothesis, it is not.

Again, if AC were less than AB,
then the ∠ABC would be less than the ∠ACB ; *Theor.* 9.
but, by hypothesis, it is not.

That is, AC is neither equal to, nor less than AB.
∴ AC is greater than AB. Q.E.D.

THEOREM 11. [Euclid I. 20]

Any two sides of a triangle are together greater than the third side.



Let ABC be a triangle.

It is required to prove that any two of its sides are together greater than the third side.

It is enough to shew that if BC is the greatest side, then BA, AC are together greater than BC.

Produce BA to D, making AD equal to AC.
Join DC.

Proof. Because AD = AC,
∴ the ∠ACD = the ∠ADC. *Theor.* 5.

But the ∠BCD is greater than the ∠ACD ;
∴ the ∠BCD is greater than the ∠ADC,
that is, than the ∠BDC.

Hence from the △BDC,
BD is greater than BC. *Theor.* 10.

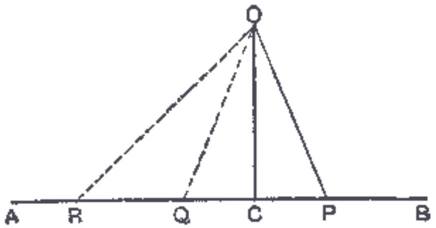
But BD = BA and AC together ;
∴ BA and AC are together greater than BC.
Q.E.D.

NOTE. This proof may serve as an exercise, but the truth of the Theorem is really self-evident. For to go from B to C along the straight line BC is clearly shorter than to go from B to A and then from A to C. In other words

The shortest distance between two points is the straight line which joins them.

THEOREM 12

Of all straight lines drawn from a given point to a given straight line the perpendicular is the least.



Let OC be the perpendicular, and OP *any* oblique, drawn from the given point O to the given straight line AB.

It is required to prove that OC *is less than* OP.

Proof. In the △OCP, since the ∠OCP is a right angle,

∴ the ∠OPC is less than a right angle ; *Theor.* 8 *Cor.*

that is, the ∠OPC is less than the ∠OCP.

∴ OC is less than OP. *Theor.* 10.

Q.E.D.

COROLLARY 1. Hence conversely, since there can be only one perpendicular and one shortest line from O to AB,

If OC *is the shortest straight line from* O *to* AB, *then* OC *is perpendicular to* AB.

COROLLARY 2. *Two obliques* OP, OQ, *which cut* AB *at equal distances from* C *the foot of the perpendicular, are equal.*

The △ˢ OCP, OCG may be shewn to be congruent by Theorem 4 ; hence OP = OQ.

COROLLARY 3. *Of two obliques* OQ, OR, *if* OR *cuts* AB *at the greater distance from* C *the foot of the perpendicular, then* OR *is greater than* OQ.

The ∠OQC is acute, ∴ the ∠OQR is obtuse ;
∴ the ∠OQR is greater than the ∠ORQ ;
∴ OR is greater than OQ.

THEOREM 13. [Euclid I. 27 and 28]

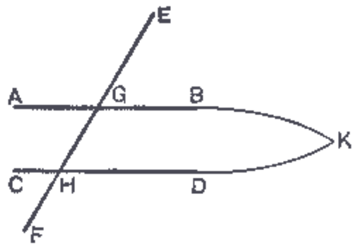
If a straight line cuts two other straight lines so as to make

(i) *the alternate angles equal,*

or (ii) *an exterior angle equal to the interior opposite angle on the same side of the cutting line,*

or (iii) *the interior angles on the same side equal to two right angles ;*

then in each case the two straight lines are parallel.



(i) Let the straight line EGHF cut the two straight lines AB, CD at G and H so as to make the alternate ∠ˢ AGH, GHD equal to one another.

It is required to prove that AB *and* CD *are parallel.*

Proof. If AB and CD are not parallel, they will meet, if produced, either towards B and D, or towards A and C.

If possible, let AB and CD, when produced, meet towards B and D, at the point K.

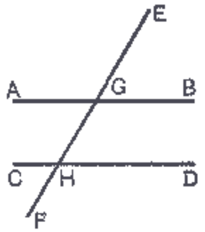
Then KGH is a triangle, of which one side KG is produced to A ;

∴ the exterior ∠AGH is greater than the interior opposite ∠GHK ; but, by hypothesis, it is not greater.

∴ AB and CD cannot meet when produced towards B and D.

Similarly it may be shewn that they cannot meet towards A and C :

∴ AB and CD are parallel.



(ii) Let the exterior ∠EGB = the interior opposite ∠GHD.

It is required to prove that AB *and* CD *are parallel.*

Proof. Because the ∠EGB = the ∠GHD,
and the ∠EGB = the vertically opposite ∠AGH ;
∴ the ∠AGH = the ∠GHD :
and these are alternate angles ;
∴ AB and CD are parallel.

(iii) Let the two interior ∠s BGH, GHD be together equal to two right angles.

It is required to prove that AB *and* CD *are parallel.*

Proof. Because the ∠s BGH, GHD together = two right angles ;
and because the adjacent ∠s BGH, AGH together = two right angles ;
∴ the ∠s BGH, AGH together = the ∠s BGH, GHD.
From these equals take the ∠BGH ;
then the remaining ∠AGH = the remaining ∠GHD :
and these are alternate angles ;
∴ AB and CD are parallel.

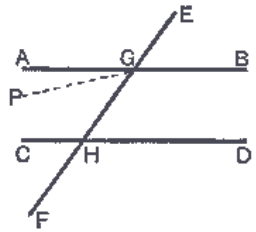
Q.E.D.

Definition. A straight line drawn across a set of given lines is called a **transversal**.

For instance, in the above diagram the line EGHF, which crosses the given lines AB, CD is a transversal.

Theorem 14. [Euclid I. 29]

If a straight line cuts two parallel lines, it makes
(i) *the alternate angles equal to one another ;*
(ii) *the exterior angle equal to the interior opposite angle on the same side of the cutting line ;*
(iii) *the two interior angles on the same side together equal to two right angles.*



Let the straight lines AB, CD be parallel, and let the straight line EGHF cut them.

It is required to prove that
(i) *the* ∠AGH = *the alternate* ∠GHD ;
(ii) *the exterior* ∠EGB = *the interior opposite* ∠GHD ;
(iii) *the two interior* ∠s BGH, GHD *together* = *two right angles.*

Proof. (i) If the ∠AGH is not equal to the ∠GHD,
suppose the ∠PGH equal to the ∠GHD, and alternate to it ;
then PG and CD are parallel. *Theor.* 13.
But, by hypothesis, AB and CD are parallel ;
∴ the two intersecting straight lines AG, PG are both parallel to CD : which is impossible. *Playfair's Axiom.*
∴ the ∠AGH is not unequal to the ∠GHD ;
that is, the alternate ∠s AGH, GHD are equal.

(ii) Again, because the ∠EGB = the vertically opposite ∠AGH ;
and the ∠AGH = the alternate ∠GHD ; *Proved.*
∴ the exterior ∠EGB = the interior opposite ∠GHD.

(iii) Lastly, the $\angle$ EGB = the $\angle$ GHD ; *Proved.*
add to each the $\angle$ BGH ;
then the $\angle^s$ EGB, BGH together = the angles BGH, GHD.
But the adjacent $\angle^s$ EGB, BGH together = two right angles ;
∴ the two interior $\angle^s$ BGH, GHD together = two right angles.

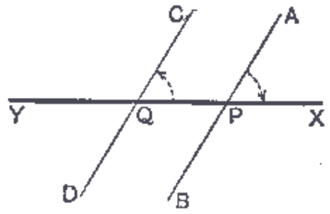
Q.E.D.

PARALLELS ILLUSTRATED BY ROTATION

The direction of a straight line is determined by the angle which it makes with some given line of reference.

Thus the direction of AB, relatively to the given line YX, is given by the angle APX.

Now suppose that AB and CD in the adjoining diagram are parallel; then we have learned that the ext. $\angle$ APX = the int.opp. $\angle$ CQX ; that is, AB and CD *make equal angles* with the line of reference YX.



This brings us to the leading idea connected with parallels :
Parallel straight lines have the same DIRECTION, *but differ in* POSITION.

The same idea may be illustrated thus :

Suppose AB to rotate about P through the $\angle$ APX, so as to take the position XY. Thence let it rotate about Q *the opposite way* through the equal $\angle$ XQC : it will now take the position CD. Thus AB may be brought into the position of CD by two rotations which, being equal and opposite, involve no final change of direction.

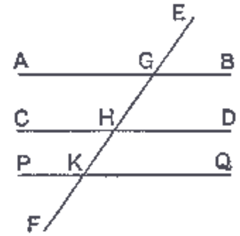
HYPOTHETICAL CONSTRUCTION. In the above diagram let AB be a fixed straight line, Q a fixed point, CD a straight line turning about Q, and YQPX any transversal through Q. Then as CD rotates, there must be one position in which the $\angle$ CQX = the fixed $\angle$ APX.

Hence through any given point we may assume a line to pass parallel to any given straight line.

Obs. If AB is a straight line, movements from A towards B, and from B towards A are said to be in opposite senses of the line AB.

THEOREM 15. [Euclid I. 30]

Straight lines which are parallel to the same straight line are parallel to one another.



Let the straight lines AB, CD be each parallel to the straight line PQ.

It is required to prove that AB *and* CD *are parallel to one another.*

Draw a straight line EF cutting AB, CD, and PQ in the points G, H, and K.

Proof. Then because AB and PQ are parallel, and EF meets them.

∴ the $\angle$ AGK = the alternate $\angle$ GKQ.

And because CD and PQ are parallel, and EF meets them,
∴ the exterior $\angle$ GHD = the interior opposite $\angle$ GKQ.

∴ the $\angle$ AGH = the $\angle$ GHD ;
and these are alternate angles ;
∴ AB and CD are parallel.

Q.E.D.

NOTE. If PQ lies between AB and CD, the Proposition needs no proof ; for it is inconceivable that two straight lines, which do not meet an intermediate straight line, should meet one another.

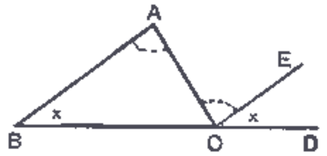
The truth of this Proposition may be readily deduced from Playfair's Axiom, of which it is the converse.

For if AB and CD were not parallel, they would meet when produced. Then there would be two intersecting straight lines both parallel to a third straight line : which is impossible.

Therefore AB and CD never meet ; that is, they are parallel.

THEOREM 16. [Euclid I. 32]

The three angles of a triangle are together equal to two right angles.



Let ABC be a triangle.

It is required to prove that the three $\angle^s$ ABC, BCA, CAB *together* = *two right angles.*

Produce BC to any point D; and suppose CE to be the line through C parallel to BA.

Proof. Because BA and CE are parallel and AC meets them,

∴ the ∠ACE = the alternate ∠CAB.

Again, because BA and CE are parallel, and BD meets them,

∴ the exterior ∠ECD = the interior opposite ∠ABC.

∴ *the whole exterior* ∠ACD = *the sum of the two interior opposite* $\angle^s$ CAB, ABC.

To each of these equals add the ∠BCA;

then the $\angle^s$ BCA, ACD together = the three $\angle^s$ BCA, CAB, ABC.

But the adjacent $\angle^s$ BCA, ACD together = two right angles.

∴ the $\angle^s$ BCA, CAB, ABC together = two right angles.

Q.E.D.

Obs. In the course of this proof the following most important property has been established.

If a side of a triangle is produced the exterior angle is equal to the sum of the two interior opposite angles.

Namely, the ext. ∠ACD = the ∠CAB + the ∠ABC.

1. *If* A, B, *and* C *denote the number of degrees in the angles of a triangle,*

   then $A + B + C = 180°$.

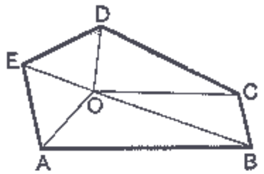
2. *If two triangles have two angles of the one respectively equal to two angles of the other, then the third angle of the one is equal to the third angle of the other.*

3. *In any right-angled triangle the two acute angles are complementary.*

4. *If one angle of a triangle is equal to the sum of the other two, the triangle is right-angled.*

5. *The sum of the angles of any quadrilateral figure is equal to four right angles.*

Corollary 1. *All the interior angles of any rectilinear figure, together with four right angles, are equal to twice as many right angles as the figure has sides.*



Let ABCDE be a rectilineal figure of n sides.

It is required to prove that all the interior angles $+ 4$ rt. $\angle^s = 2n$ rt. $\angle^s$.

Take any point O within the figure, and join O to each of its vertices.

Then the figure is divided into n triangles.

And the three $\angle^s$ of each $\triangle$ together $= 2$ rt. $\angle^s$.
Hence *all* the $\angle^s$ of *all* the $\triangle^s$ together $= 2n$ rt. $\angle^s$.

But all the $\angle^s$ of all the $\triangle^s$ make up all the interior angles of the figure together with the angles at O, which $= 4$ rt. $\angle^s$.

$\therefore$ all the int. $\angle^s$ of the figure $+ 4$ rt. $\angle^s = 2n$ rt. $\angle^s$.

Q.E.D.

Definition. A **regular** polygon is one which has all its sides equal and all its angles equal.

Thus if D denotes the number of degrees in each angle of a *regular* polygon of n sides, the above result may be stated thus:

$$n\text{D} + 360^\circ = n \, . \, 180^\circ.$$

EXAMPLE

Find the number of degrees in each angle of
(i) a regular hexagon (6 sides);
(ii) a regular octagon (8 sides);
(iii) a regular decagon (10 sides).

Corollary 2. *If the sides of a rectilineal figure, which has no reflex angle, are produced in order, then all the exterior angles so formed are together equal to four right angles.*

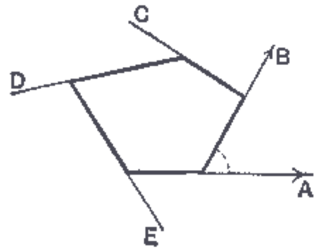
1st Proof. Suppose, as before, that the figure has n sides; and consequently n vertices.

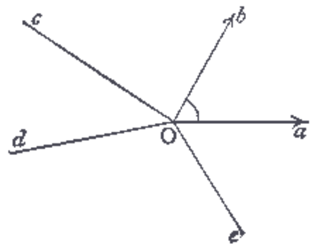
Now at *each* vertex
the interior $\angle$ + the exterior $\angle = 2$ rt. $\angle^s$;
and there are n vertices,
$\therefore$ the sum of the int. $\angle^s$ + the sum of the ext. $\angle^s = 2n$ rt. $\angle^s$.

But by Corollary 1,
the sum of the int. $\angle^s + 4$ rt. $\angle^s = 2n$ rt. $\angle^s$;
$\therefore$ the sum of the ext. $\angle^s = 4$ rt. $\angle^s$.

Q.E.D.

2nd Proof.





Take any point O, and suppose Oa, Ob, Oc, Od, and Oe, are lines parallel to the sides marked, A, B, C, D, E (and drawn from O in the *sense* in which those sides were produced).

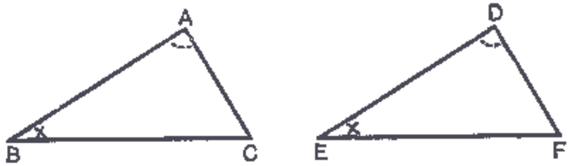
Then the exterior $\angle$ between the sides A and B = the $\angle a$Ob.

And the other exterior $\angle^s$ = the $\angle^s$ bOc, cOd, dOe, eOa, respectively.

$\therefore$ the sum of the ext. $\angle^s$ = the sum of the $\angle^s$ at O
$= 4$ rt. $\angle^s$.

Theorem 17. [Euclid I. 26]

If two triangles have two angles of one equal to two angles of the other, each to each, and any side of the first equal to the corresponding side of the other, the triangles are equal in all respects.



Let ABC, DEF be two triangles in which

the ∠A = the ∠D,

the ∠B = the ∠E,

also let the side BC = the corresponding side EF.

It is required to prove that the △^{s} ABC, DEF *are equal in all respects.*

Proof. The sum of the ∠^{s} A, B, and C

= 2 rt. ∠^{s} *Theor.* 16.

= the sum of the ∠^{s} D, E, and F;

and the ∠^{s} A and B = the ∠^{s} D and E respectively,

∴ the ∠C = the ∠F.

Apply the △ABC to the △DEF, so that B falls on E, and BC along EF.

Then because BC = EF,

∴ C must coincide with F.

And because the ∠B = the ∠E,

∴ BA must fall along ED.

And because the ∠C = the ∠F,

∴ CA must fall along FD.

∴ the point A, which falls both on ED and on FD, must coincide with D, the point in which these lines intersect.

∴ the △ABC coincides with the △DEF,

and is therefore equal to it in all respects.

So that AB = DE, and AC = DF;

and the △ABC = the △DEF in area. Q.E.D.

Three cases of the congruence of triangles have been dealt with in Theorems 4, 7, 17, the results of which may be summarised as follows:

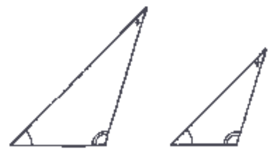
Two triangles are equal in all respects when the following three parts in each are severally equal:

1. *Two sides, and the included angle.* *Theorem* 4.
2. *The three sides.* *Theorem* 7.
3. *Two angles and one side, the side given in one triangle* CORRESPONDING *to that given in the other.* *Theorem* 17.

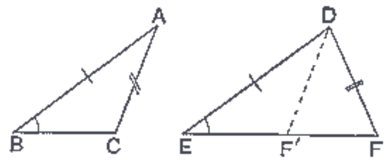
Two triangles are not, however, necessarily equal in all respects when *any three parts* of one are equal to the corresponding parts of the other.

For example:

(i) When the *three angles* of one are equal to the *three angles* of the other, each to each, the adjoining diagram shews that the triangles need not be equal in all respects.



(ii) When *two sides and one angle* in one are equal to *two sides and one angle* of the other, the given angles being *opposite* to equal sides, the diagram below shews that the triangles need not be equal in all respects.



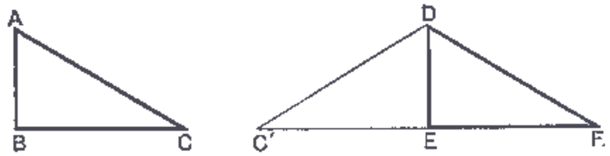
For if AB = DE, and AC = DF, and the ∠ABC = the ∠DEF, it will be seen that the shorter of the given sides in the triangle DEF may lie in either of the positions DF or DF′.

Note. From these data it may be shewn that the angles opposite to the equal sides AB, DE are either *equal* (as for instance the ∠^{s} ACB, DF′E) or *supplementary* (as the ∠^{s} ACB, DFE); and that in the former case the triangles are equal in all respects. This is called the **ambiguous case** in the congruence of triangles. [See Problem 9, p. 82.]

If the given angles at B and E are right angles, the ambiguity disappears. This exception is proved in the following Theorem.

Theorem 18

Two **right-angled** *triangles which have their hypotenuses equal, and one side of one equal to one side of the other, are equal in all respects.*



Let ABC, DEF be two **right-angled** triangles, in which
the $\angle^s$ ABC, DEF are right angles,
the hypotenuse AC = the hypotenuse DF,
and AB = DE.

It is required to prove that the $\triangle^s$ ABC, DEF *are equal in all respects.*

Proof. Apply the $\triangle$ABC to the $\triangle$DEF, so that AB falls on the equal line DE, and C on the side of DE opposite to F.

Let C′ be the point on which C falls.

Then DEC′ represents the $\triangle$ABC in its new position.

Since each of the $\angle^s$ DEF, DEC′ is a right angle,
$\therefore$ EF and EC′ are in one straight line.

And in the $\triangle$C′DF, because DF = DC′ (*i.e.* AC),
$\therefore$ the $\angle$DFC′ = the $\angle$DC′F. *Theor.* 5.

Hence in the $\triangle^s$ DEF, DEC′,

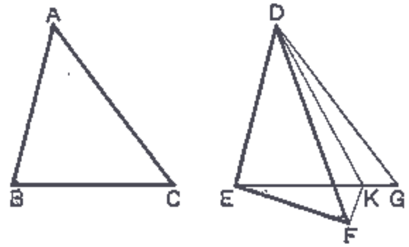
because { the $\angle$DEF = the $\angle$DEC′, being right angles ;
the $\angle$DFE = the $\angle$DC′E, *Proved.*
and the side DE is common.

$\therefore$ the $\triangle^s$ DEF, DEC′ are equal in all respects ; *Theor.* 17.
that is, the $\triangle^s$ DEF, ABC are equal in all respects.

Q.E.D.

*Theorem 19. [Euclid I. 24]

If two triangles have two sides of the one equal to two sides of the other, each to each, but the angle included by the two sides of one greater than the angle included by the corresponding sides of the other ; then the base of that which has the greater angle is greater than the base of the other.



Let ABC, DEF be two triangles, in which
BA = ED,
and AC = DF,
but the $\angle$BAC is greater than the $\angle$EDF.

It is required to prove that the base BC *is greater than the base* EF.

Proof. Apply the $\triangle$ABC to the $\triangle$DEF, so that A falls on D, and AB along DE.

Then because AB = DE, B must coincide with E.

Let DG, GE represent AC, CB in their new position.

Then if EG passes through F, EG is greater than EF ;
that is, BC is greater than EF.

But if EG does not pass through F, suppose that DK bisects the $\angle$FDG, and meets EG in K. Join FK.

Then in the $\triangle^s$ FDK, GDK,

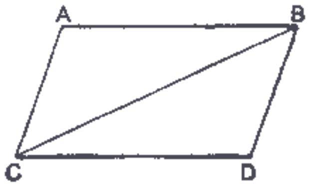
because { FD = GD,
DK is common to both,
and the included $\angle$FDK = the included $\angle$GDK ;

$\therefore$ FK = GK. *Theor.* 4.

Now that the two sides EK, KF are greater than EF ;
that is, EK, KG are greater than EF.
$\therefore$ EG (or BC) is greater than EF. Q.E.D.

THEOREM 20. [Euclid I. 33]

The straight lines which join the extremities of two equal and parallel straight lines towards the same parts are themselves equal and parallel.



Let AB and CD be equal and parallel straight lines; and let them be joined towards the same parts by the straight lines AC and BD.

It is required to prove that AC *and* BD *are equal and parallel.*

Join BC.

Proof. Then because AB and CD are parallel, and BC meets them,

∴ the ∠ABC = the alternate ∠DCB.

Now in the $\triangle^s$ ABC, DCB,

because { AB = DC,
BC is common to both;
and the ∠ABC = the ∠DCB; *Proved.*

∴ the triangles are equal in all respects;

so that AC = DB, ...(i)

and the ∠ACB = ∠DBC.

But these are alternate angles;

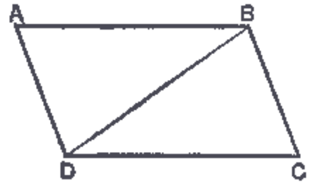
∴ AC and BD are parallel. ....(ii)

That is, AC and BD are both equal and parallel.

Q.E.D.

THEOREM 21. [Euclid I. 34]

The opposite sides and angles of a parallelogram are equal to one another, and each diagonal bisects the parallelogram.



Let ABCD be a parallelogram, of which BD is a diagonal.

It is required to prove that

(i) AB = CD, *and* AD = CB,
(ii) *the* ∠BAD = *the* ∠DCB,
(iii) *the* ∠ADC = *the* ∠CBA,
(iv) *the* △ABD = *the* △CDB *in area.*

Proof. Because AB and DC are parallel, and BD meets them,

∴ the ∠ABD = the alternate ∠CDB.

Again, because AD and BC are parallel, and BD meets them,

∴ the ∠ADB = the alternate ∠CBD.

Hence in the $\triangle^s$ ABD, CDB,

because { the ∠ABD = the ∠CDB,
the ∠ADB = the ∠CBD, *Proved.*
and BD is common to both;

∴ the triangles are equal in all respects; *Theor.* 17.

so that AB = CD, and AD = CB; ....(i)

and the ∠BAD = the ∠DCB; ....(ii)

and the △ABD = the △CDB in area. ....(iv)

And because the ∠ADB = the ∠CBD, *Proved.*

and the ∠CDB = the ∠ABD,

∴ the whole ∠ADC = the whole ∠CBA. ......(iii)

Q.E.D.

COROLLARY 1. *If one angle of a parallelogram is a right angle, all its angles are right angles.*

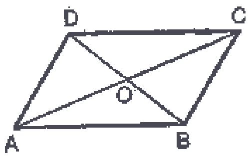
In other words :

All *the angles of a rectangle are right angles.*

For the sum of two consecutive ∠s = 2 rt. ∠s ; (*Theor.* 14.)
∴, if one of these is a rt. angle, the other must be a rt. angle.
And the opposite angles of the par^m are equal ;
∴ all the angles are right angles.

COROLLARY 2. *All the sides of a square are equal ; and all its angles are right angles.*

COROLLARY 3. *The diagonals of a parallelogram bisect one another.*



Let the diagonals AC, BD of the par^m ABCD intersect at O.

To prove AO = OC, *and* BO = OD.

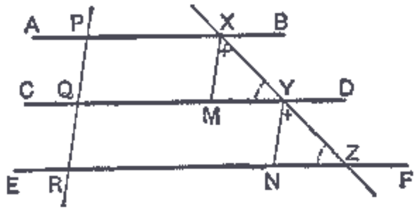
In the △s AOB, COD,

because { the ∠OAB = the alt. ∠OCD,
the ∠AOB = vert. opp. ∠COD,
and AB = the opp. side CD ;

∴ OA = OC ; and OB = OD. *Theor.* 17.

THEOREM 22

If there are three or more parallel straight lines, and the intercepts made by them on any transversal are equal, then the corresponding intercepts on any other transversal are also equal.



Let the parallels AB, CD, EF cut off equal intercepts PQ, QR from the transversal PQR ; and let XY, YZ be the corresponding intercepts cut off from any other transversal XYZ.

It is required to prove that XY = YZ.

Through X and Y let XM and YN be drawn parallel to PR.

Proof. Since CD and EF are parallel, and XZ meets them,
∴ the ∠XYM = the corresponding ∠YZN.
And since XM, YN are parallel, each being parallel to PR,
∴ the ∠MXY = the corresponding ∠NYZ.

Now the figures PM, QN are parallelograms,
∴ XM = the opp. side PQ, and YN = the opp. side QR ;
and since by hypothesis PQ = QR,
∴ XM = YN.

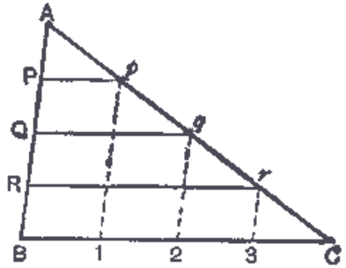
Then in the △s XMY, YNZ,

because { the ∠XYM = the ∠YZN,
the ∠MXY = the ∠NYZ,
and XM = YN ;

∴ the triangles are identically equal ; *Theor.* 17.
∴ XY = YZ.

Q.E.D.

Corollary. *In a triangle ABC, if a set of lines Pp, Qq, Rr, ..., drawn parallel to the base, divide one side AB into equal parts, they also divide the other side AC into equal parts.*



The lengths of the parallels P*p*, Q*q*, R*r*, ..., may thus be expressed in terms of the base BC.

Through *p*, *q*, and *r* let *p*1, *q*2, *r*3 be drawn par[l] to AB.

Then, by Theorem 22, these par[ls] divide BC into *four* equal parts, of which P*p* evidently contains *one*, Q*q* *two*, and R*r* *three*.

In other words,

$$Pp=\frac{1}{4}\cdot BC;\quad Qq=\frac{2}{4}\cdot BC;\quad Rr=\frac{3}{4}\cdot BC.$$

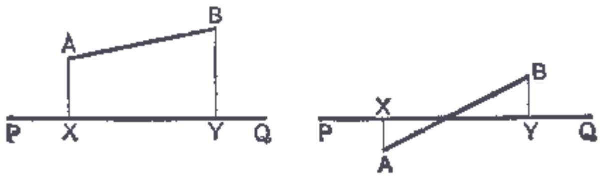
Similarly if the given par[ls] divide AB into *n* equal parts,

$$Pp=\frac{1}{n}\cdot BC,\quad Qq=\frac{2}{n}\cdot BC,\quad Rr=\frac{3}{n}\cdot BC;\ \text{and so on.}$$

*** *Problem 7, p. 78, should now be worked.*

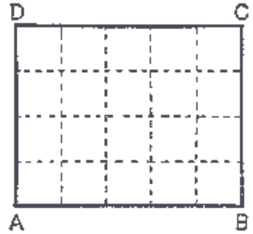
DEFINITION

If from the extremities of a straight line AB perpendiculars AX, BY are drawn to a straight line PQ of indefinite length, then XY is said to be the **orthogonal projection** of AB on PQ.



Theorem 23

Area of a rectangle. *If the number of units in the length of a rectangle is multiplied by the number of units in its breadth, the product gives the number of square units in the area.*



Let ABCD represent a rectangle whose length AB is 5 cm., and whose breadth AD is 4 cm.

Divide AB into 5 equal parts, and BC into 4 equal parts, and through the points of division of each line draw parallels to the other.

The rectangle ABCD is now divided into compartments, each of which represents one square centimetre.

Now there are 4 rows, each containing 5 squares,

∴ the rectangle contains 5×4 square centimetres.

Similarly, if the length $=a$ linear units, and the breadth $=b$ linear units

the rectangle contains ab *units of* area.

And if each side of a square $=a$ linear units,

the square contains a^2 *units of* area.

These statements may be thus abridged:

the area of a rectangle = length × breadth ....(i),

the area of a square = (side)^2 ....(ii).

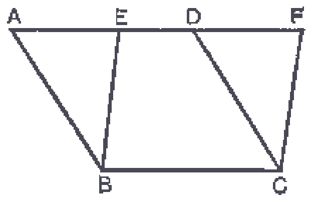
Q.E.D.

Corollaries. (i) *Rectangles which have equal lengths and equal breadths have equal areas.*

(ii) *Rectangles which have equal areas and equal lengths have also equal breadths.*

Theorem 24. [Euclid I. 35]

Parallelograms on the same base and between the same parallels are equal in area.



Let the par^{ms} ABCD, EBCF be on the same base BC, and between the same par^{ls} BC, AF.

It is required to prove that

the par^m ABCD = *the par^m* EBCF *in area.*

Proof. In the $\triangle^s$ FDC, EAB,

because { DC = the opp. side AB ; *Theor.* 21.
the ext. $\angle$FDC = the int. opp. $\angle$EAB ; *Theor.* 14.
the int. $\angle$DFC = the ext. $\angle$AEB ; }

$\therefore$ the $\triangle$FDC = the $\triangle$EAB. *Theor.* 17.

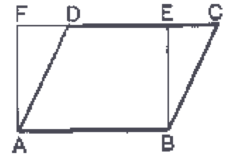
Now, if from the whole fig. ABCF the $\triangle$FDC is taken, the remainder is the par^m ABCD.

And if from the whole fig. ABCF the $\triangle$EAB is taken, the remainder is the par^m EBCF.

$\therefore$ these remainders are equal ;

that is, the par^m ABCD = the par^m EBCF. Q.E.D.

The Area of a Parallelogram



Let ABCD be a parallelogram, and ABEF the rectangle on the same base AB and of the same altitude BE. Then by Theorem 24,

area of par^m ABCD = area of rect. ABEF
= AB × BE
= *base* × *altitude.*

Corollary. Since the area of a parallelogram depends only on its base and altitude, it follows that

Parallelograms on equal bases and of equal altitudes are equal in area.

THE AREA OF A TRIANGLE

If BC and AF respectively contain a units and p units of length, the rectangle BDEC contains ap units of area.

$\therefore$ the area of the $\triangle$ABC = $\frac{1}{2}ap$ units of area.

This result may be stated thus :

Area of a Triangle = $\frac{1}{2}$. *base* × *altitude.*

THEOREM 25

The Area of a Triangle. *The area of a triangle is half the area of the rectangle on the same base and having the same altitude.*

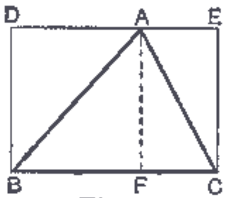


Fig. 1.

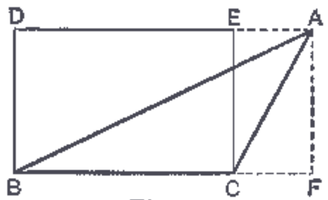


Fig. 2.

Let ABC be a triangle, and BDEC a rectangle on the same base BC and with the same altitude AF.

It is required to prove that the △ABC *is half the rectangle* BDEC.

Proof. Since AF is perp. to BC, each of the figures DF, EF is a rectangle.

Because the diagonal AB bisects the rectangle DF,
∴ the △ABF is half the rectangle DF.

Similarly, the △AFC is half the rectangle FE.

∴ adding these results in Fig. 1, and taking the difference in Fig. 2,
the △ABC is half the rectangle BDEC.

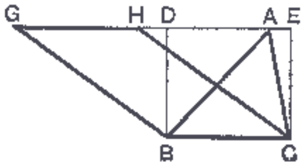
Q.E.D.

COROLLARY. *A triangle is half any parallelogram on the same base and between the same parallels.*

For the △ABC is half the rect. BCED.

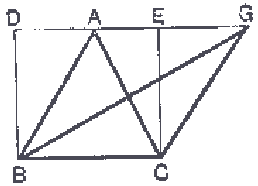
And the rect. BCED = any parm BCHG on the same base and between the same parls.

∴ the △ABC is half the parm BCHG.



THEOREM 26. [Euclid I. 37]

Triangles on the same base and between the same parallels (hence, of the same altitude) are equal in area.



Let the △s ABC, GBC be on the same base BC and between the same parls BC, AG.

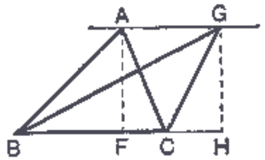
It is required to prove that the △ABC = *the* △GBC *in area.*

Proof. If BCED is the rectangle on the base BC, and between the same parallels as the given triangles,
the △ABC is half the rect. BCED; *Theor.* 25.
also the △GBC is half the rect. BCED;
∴ the △ABC = the △GBC. Q.E.D.

Similarly, *triangles on equal bases and of equal altitudes are equal in area.*

THEOREM 27. [Euclid I. 39]

If two triangles are equal in area, and stand on the same base and on the same side of it, they are between the same parallels.



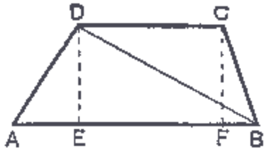
Let the △s ABC, GBC, standing on the same base BC, be equal in area; and let AF and GH be their altitudes.

It is required to prove that AG *and* BC *are parl.*

Proof. The △ABC is half the rectangle contained by BC and AF;
and the △GBC is half the rectangle contained by BC and GH;
∴ the rect. BC, AF = the rect. BC, GH;
∴ AF = GH. *Theor.* 23, *Cor.* 2.
Also AF and GH are parl;
hence AG and FH, that is BC, are parl. Q.E.D.

THEOREM 28

To find the area of (i) *a trapezium.*
(ii) *any quadrilateral.*



(i) Let ABCD be a trapezium, having the sides AB, CD parallel. Join BD, and from C and D draw perpendiculars CF, DE to AB.

Let the parallel sides AB, CD measure a and b units of length, and let the height CF contain h units.

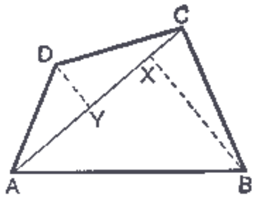
Then the area of ABCD = △ABD + △DBC

$$=\frac{1}{2}\text{AB}\,.\,\text{DE}+\frac{1}{2}\text{DC}\,.\,\text{CF}$$

$$=\frac{1}{2}ah+\frac{1}{2}bh=\frac{h}{2}(a+b).$$

That is,

the area of a trapezium $=\frac{1}{2}$ *height* × (*the sum of the parallel sides*).



(ii) Let ABCD be any quadrilateral. Draw a diagonal AC; and from B and D draw perpendiculars BX, DY to AC. These perpendiculars are called **offsets.**

If AC contains d units of length, and BX, DY p and q units respectively,

the area of the quad[l] ABCD = △ABC + △ADC

$$=\frac{1}{2}\text{AC}\,.\,\text{BX}+\frac{1}{2}\text{AC}\,.\,\text{DY}$$

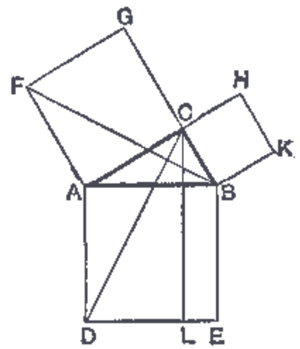
$$=\frac{1}{2}dp+\frac{1}{2}dq=\frac{1}{2}d(p+q).$$

That is to say,

the area of a quadrilateral $=\frac{1}{2}$ *diagonal* × (*sum of offsets*).

THEOREM 29. [Euclid I. 47]

In a right-angled triangle the square described on the hypotenuse is equal to the sum of the squares described on the other two sides.



Let ABC be a right-angled △, having the angle ACB a rt. ∠.

It is required to prove that the square on the hypotenuse AB = *the sum of the squares on* AC, CB.

On AB describe the sq. ADEB; and on AC, CB describe the sqq. ACGF, CBKH.

Through C draw CL par[l] to AD or BE.
Join CD, FB.

Proof. Because each of the ∠s ACB, ACG is a rt. ∠,
∴ BC and CG are in the same st. line.

Now the rt. ∠BAD = the rt. ∠FAC;
add to each the ∠CAB:
then the whole ∠CAD = the whole ∠FAB.

Then in the △s CAD, FAB,

because { CA = FA,
AD = AB,
and the included ∠CAD = the included ∠FAB;

∴ the △CAD = the △FAB. *Theor.* 4.

Now the rect. AL is double of the $\triangle$CAD, being on the same base AD, and between the same par[ls] AD, CL.

And the sq. GA is double of the $\triangle$FAB, being on the same base FA, and between the same par[ls] FA, GB.

$\therefore$ the rect. AL = the sq. GA.

Similarly by joining CE, AK, it can be shewn that

the rect. BL = the sq. HB.

$\therefore$ the whole sq. AE = the sum of the sqq. GA, HB :

that is, the square on the hypotenuse AB = the sum of the squares on the two sides AC, CB.

Q.E.D.

Obs. This is known as the Theorem of Pythagoras. The result established may be stated as follows :

$$AB^2 = BC^2 + CA^2.$$

That is, if a and b denote the lengths of the sides containing the right angle; and if c denotes the hypotenuse,

$$c^2 = a^2 + b^2.$$

Hence $\qquad a^2 = c^2 - b^2$; and $b^2 = c^2 - a^2$.

NOTE 1. The following important results should be noticed.

If CL and AB intersect in O, it has been shewn in the course of the proof that

the sq. GA = the rect. AL ;
that is, AC^2 = the rect. contained by AB, AO. ....(i)

Also the sq. HB = the rect. BL ;
that is, BC^2 = the rect. contained by BA, BO. ....(ii)

NOTE 2. It can be proved by superposition that *squares standing on equal sides are equal in area.*

Hence we conclude, conversely,

If two squares are equal in area they stand on equal sides.

EXPERIMENTAL PROOFS OF PYTHAGORAS'S THEOREM

I. Here ABC is the given rt.-angled $\triangle$; and ABED is the square on the hypotenuse AB.

By drawing lines par[l] to the sides BC, CA, it is easily seen that the sq. BD is divided into 4 rt.-angled $\triangle$s, each identically equal to ABC, together with a central square.

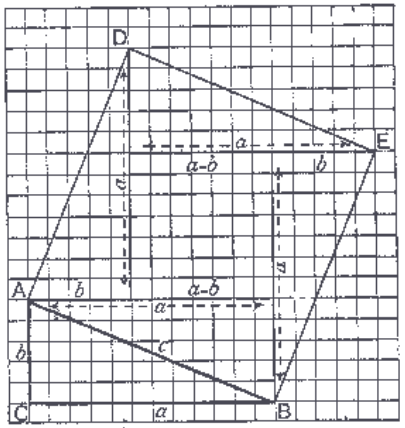
Hence

sq. on hypotenuse c = 4 rt. $\angle$[d] $\triangle$s + the central square

$= 4 \cdot \frac{1}{2}ab + (a-b)^2$

$= 2ab + a^2 - 2ab + b^2$

$= a^2 + b^2$.

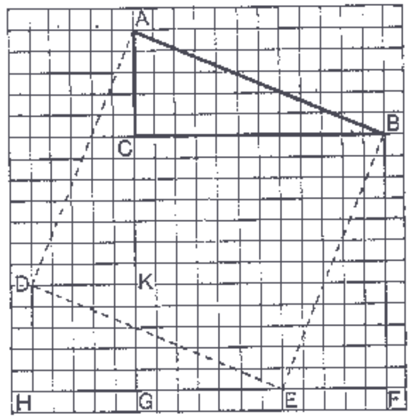


II. Here ABC is the given rt.-angled $\triangle$, and the figs. CF, HK are the sqq. on CB, CA placed side by side.

FE is made equal to DH or CA ; and the two sqq. CF, HK are cut along the lines BE, ED.

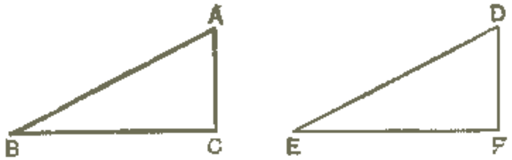
Then it will be found that the $\triangle$DHE may be placed so as to fill up the space ACB ; and the $\triangle$BFE may be made to fill the space AKD.

Hence the two sqq. CF, HK may be fitted together so as to form the single fig. ABED, which will be found to be a perfect square, namely the square on the hypotenuse AB.



THEOREM 30. [Euclid I. 48]

If the square described on one side of a triangle is equal to the sum of the squares described on the other two sides, then the angle contained by these two sides is a right angle.



Let ABC be a triangle in which
the sq. on AB = the sum of the sqq. on BC, CA.

It is required to prove that ACB *is a right angle.*

Make EF equal to BC.
Draw FD perp^r to EF, and make FD equal to CA.
Join ED.

Proof. Because EF = BC,
∴ the sq. on EF = the sq. on BC.
And because FD = CA,
∴ the sq. on FD = the sq. on CA.
Hence the sum of the sqq. on EF, FD = the sum of the sqq. on BC, CA.
But since EFD is a rt. ∠,
∴ the sum of the sqq. on EF, FD = the sq. on DE : *Theor.* 29.
And, by hypothesis, the sqq. on BC, CA = the sq. on AB.
∴ the sq. on DE = the sq. on AB.
∴ DE = AB.

Then in the △s ACB, DFE,

because { AC = DF,
CB = FE,
and AB = DE ;

∴ the ∠ACB = the ∠DFE. *Theor.* 7.

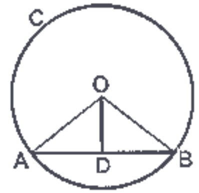
But, by construction, DFE is a right angle ;
∴ the ∠ACB is a right angle.

Q.E.D.

THEOREM 31. [Euclid III. 3]

If a straight line drawn from the centre of a circle bisects a chord which does not pass through the centre, it cuts the chord at right angles.

Conversely, if it cuts the chord at right angles, it bisects it.



Let ABC be a circle whose centre is O ; and let OD bisect a chord AB which does not pass through the centre.

It is required to prove that OD *is perp. to* AB.

Join OA, OB.

Proof. Then in the △s ADO, BDO,

because { AD = BD, by hypothesis,
OD is common,
and OA = OA, being radii of the circle ;

∴ the ∠ADO = the ∠BDO ; *Theor.* 7.
and these are adjacent angles,
∴ OD is perp. to AB. Q.E.D.

Conversely. Let OD be perp. to the chord AB.

It is required to prove that OD *bisects* AB.

Proof. In the △s ODA, ODB,

because { the ∠s ODA, ODB are *right angles*,
the hypotenuse OA = the hypotenuse OB,
and OD is common ;

∴ DA = DB ; *Theor.* 18.
that is, OD bisects AB at D. Q.E.D.

Corollary 1. *The straight line which bisects a chord at right angles passes through the centre.*

Corollary 2. *A straight line cannot meet a circle at more than two points.*

For suppose a st. line meets a circle whose centre is O at the points A and B.

Draw OC perp. to AB.

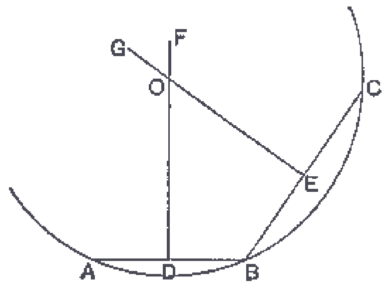
Then AC = CB.

Now if the circle were to cut AB in a third point D, AC would also be equal to CD, which is impossible.

Corollary 3. *A chord of a circle lies wholly within it.*

Theorem 32

One circle, and only one, can pass through any three points not in the same straight line.



Let A, B, C be three points not in the same straight line.

It is required to prove that one circle, and only one, can pass through A, B, *and* C.

Join AB, BC,

Let AB and BC be bisected at right angles by the lines DF, EG.

Then since AB and BC are not in the same st. line, DF and EG are not par[l].

Let DF and EG meet in O.

Proof. Because DF bisects AB at right angles,

∴ every point on DF is equidistant from A and B.

Prob. 14.

Similarly every point on EG is equidistant from B and C.

∴ O, the only point common to DF and EG, is equidistant from A, B, and C;

and there is no other point equidistant from A, B, and C.

∴ a circle having its centre at O and radius OA will pass through B and C; and this is the only circle which will pass through the three given points. Q.E.D.

Corollary 1. *The size and position of a circle are fully determined if it is known to pass through three given points;* for then the position of the centre and length of the radius can be found.

Corollary 2. *Two circles cannot cut one another in more than two points without coinciding entirely;* for if they cut at three points they would have the same centre and radius.

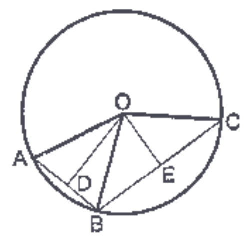
Hypothetical Construction. *From Theorem 32 it appears that we may suppose a circle to be drawn through any three points not in the same straight line.*

For example, a circle can be assumed to pass through the vertices of any triangle.

Definition. The circle which passes through the vertices of a triangle is called its **circum-circle**, and is said to be **circumscribed** about the triangle. The centre of the circle is called the **circum-centre** of the triangle, and the radius is called the **circum-radius.**

* Theorem 33. [Euclid III. 9]

If from a point within a circle more than two equal straight lines can be drawn to the circumference, that point is the centre of the circle.



Let ABC be a circle, and O a point within it from which more than two equal st. lines are drawn to the ○ce, namely OA, OB, OC.

It is required to prove that O *is the centre of the circle* ABC.

Join AB, BC.

Let D and E be the middle points of AB and BC respectively.

Join OD, OE.

Proof. In the △s ODA, ODB,

because { DA = DB,
DO is common,
and OA = OB, by hypothesis;

∴ the ∠ODA = the ∠ODB; *Theor.* 7.

∴ these angles, being adjacent, are rt. ∠s.

Hence DO bisects the chord AB at right angles, and therefore passes through the centre. *Theor.* 31, *Cor.* 1.

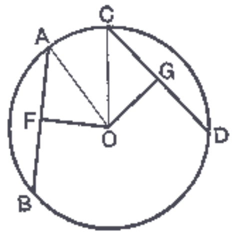
Similarly it may be shewn that EO passes through the centre.

∴ O, which is the only point common to DO and EO, must be the centre. Q.E.D.

THEOREM 34. [Euclid III. 14]

Equal chords of a circle are equidistant from the centre.

Conversely, chords which are equidistant from the centre are equal.



Let AB, CD be chords of a circle whose centre is O, and let OF, OG be perpendiculars on them from O.

First. Let AB = CD.

It is required to prove that AB *and* CD *are equidistant from* O.

Join OA, OC.

Proof. Because OF is perp. to the chord AB,

∴ OF bisects AB ; *Theor.* 31.

∴ AF is half of AB.

Similarly CG is half of CD.

But, by hypothesis, AB = CD,

∴ AF = CG.

Now in the $\triangle^s$ OFA, OGC,

because { the $\angle^s$ OFA, OGC are *right angles*, the hypotenuse OA = the hypotenuse OC, and AF = CG ;

∴ the triangles are equal in all respects ; *Theor.* 18.

so that OF = OG ;

that is, AB and CD are equidistant from O.

Q.E.D.

Conversely. Let OF = OG.

It is required to prove that AB = CD.

Proof. As before it may be shewn that AF is half of AB, and CG half of CD.

Then in the $\triangle^s$ OFA, OGC,

because { the $\angle^s$ OFA, OGC are *right angles*, the hypotenuse OA = the hypotenuse OC, and OF = OG ;

∴ AF = CG ; *Theor.* 18.

∴ the doubles of these are equal ;

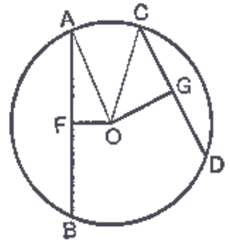
that is, AB = CD.

Q.E.D.

THEOREM 35. [Euclid III. 15]

Of any two chords of a circle, that which is nearer to the centre is greater than one more remote.

Conversely, the greater of two chords is nearer to the centre than the less.



Let AB, CD be chords of a circle whose centre is O, and let OF, OG be perpendiculars on them from O.

It is required to prove that

(i) *if* OF *is less than* OG, *then* AB *is greater than* CD ;

(ii) *if* AB *is greater than* CD, *then* OF *is less than* OG.

Join OA, OC.

Proof. Because OF is perp. to the chord AB,

∴ OF bisects AB ;

∴ AF is half of AB.

Similarly CG is half of CD.

Now OA = OC,

∴ the sq. on OA = the sq. on OC.

But since the ∠OFA is a rt. angle,

∴ the sq. on OA = the sqq. on OF, FA.

Similarly the sq. on OC = the sqq. on OG, GC.

∴ the sqq. on OF, FA = the sqq. on OG, GC.

(i) Hence if OF is given less than OG ;

the sq. on OF is less than the sq. on OG.

∴ the sq. on FA is greater than the sq. on GC ;

∴ FA is greater than GC :

∴ AB is greater than CD.

(ii) But if AB is given greater than CD,

that is, if FA is greater than GC ;

then the sq. on FA is greater than the sq. on GC.

∴ the sq. on OF is less than the sq. on OG ;

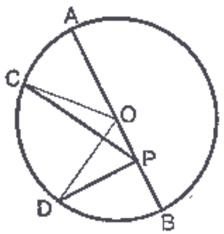
∴ OF is less than OG. Q.E.D.

COROLLARY. *The greatest chord in a circle is a diameter.*

* THEOREM 36. [Euclid III. 7]

If from any internal point, not the centre, straight lines are drawn to the circumference of a circle, then the greatest is that which passes through the centre, and the least is the remaining part of that diameter.

And of any other two such lines the greater is that which subtends the greater angle at the centre.



Let ACDB be a circle, and from P any internal point, which is not the centre, let PA, PB, PC, PD be drawn to the ○ce, so that PA passes through the centre O, and PB is the remaining part of that diameter. Also let the ∠POC at the centre subtended by PC be greater than the ∠POD subtended by PD.

It is required to prove that of these st. lines

(i) PA *is the greatest,*

(ii) PB *is the least,*

(iii) PC *is greater than* PD.

Join OC, OD.

Proof. (i) In the △POC, the sides PO, OC are together greater than PC. *Theor.* 11.

But OC = OA, being radii;

∴ PO, OA are together greater than PC;

that is, PA is greater than PC.

Similarly PA may be shewn to be greater than any other st. line drawn from P to the ○ce;

∴ PA is the greatest of all such lines.

(ii) In the △OPD, the sides OP, PD are together greater than OD.

But OD = OB, being radii;

∴ OP, PD are together greater than OB.

Take away the common part OP;

then PD is greater than PB.

Similarly any other st. line drawn from P to the ○ce may be shewn to be greater than PB;

∴ PB is the least of all such lines.

(iii) In the △s POC, POD,

because { PO is common,
OC = OD, being radii,
but the ∠POC is greater than the ∠POD;

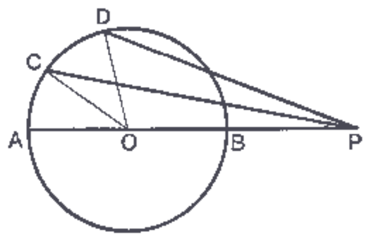
∴ PC is greater than PD. *Theor.* 19.

Q.E.D.

*THEOREM 37. [Euclid III. 8]

If from any external point straight lines are drawn to the circumference of a circle, the greatest is that which passes through the centre, and the least is that which when produced passes through the centre.

And of any other two such lines, the greater is that which subtends the greater angle at the centre.



Let ACDB be a circle, and from any external point P let the lines PBA, PC, PD be drawn to the ○ce, so that PBA passes through the centre O, and so that the ∠POC subtended by PC at the centre is greater than the ∠POD subtended by PD.

It is required to prove that of these st. lines

(i) PA *is the greatest,*

(ii) PB *is the least,*

(iii) PC *is greater than* PD.

Join OC, OD.

Proof. (i) In the △POC, the sides PO, OC are together greater than PC.

But OC = OA, being radii ;

∴ PO, OA are together greater than PC ;

that is, PA is greater than PC.

Similarly PA may be shewn to be greater than any other st. line drawn from P to the ○ce ;

that is, PA is the greatest of all such lines.

(ii) In the △POD, the sides PD, DO are together greater than PO.

But OD = OB, being radii ;

∴ the remainder PD is greater than the remainder PB.

Similarly any other st. line drawn from P to the ○ce may be shewn to be greater than PB ;

that is, PB is the least of all such lines.

(iii) In the △s POC, POD,

because { PO is common,
OC = OD, being radii ;
but the ∠POC is greater than the ∠POD ;

∴ PC is greater than PD. *Theor.* 19.

Q.E.D.

Theorem 38. [Euclid III. 20]

The angle at the centre of a circle is double of an angle at the circumference standing on the same arc.

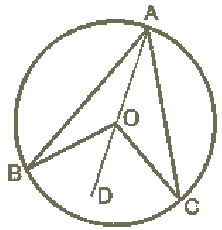


Fig. 1.

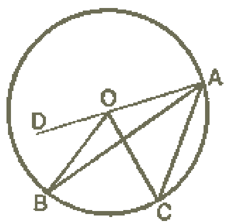


Fig. 2.

Let ABC be a circle, of which O is the centre; and let BOC be the angle at the centre, and BAC an angle at the $\odot^{ce}$, standing on the same arc BC.

It is required to prove that the ∠BOC *is twice the* ∠BAC.

Join AO, and produce it to D.

Proof. In the △OAB, because OB = OA,

∴ the ∠OAB = the ∠OBA.

∴ the sum of the ∠s OAB, OBA = twice the ∠OAB.

But the ext. ∠BOD = the sum of the ∠s OAB, OBA;

∴ the ∠BOD = twice the ∠OAB.

Similarly the ∠DOC = twice the ∠OAC.

∴, adding these results in Fig. 1, and taking the difference in Fig. 2, it follows in each case that

the ∠BOC = twice the ∠BAC. Q.E.D.

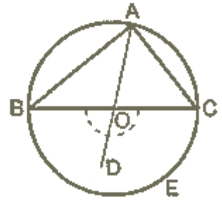


Fig. 3.

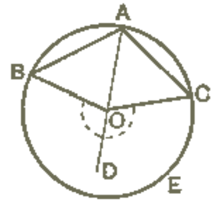


Fig. 4.

Obs. If the arc BEC, on which the angles stand, is a semi-circumference, as in Fig. 3, the ∠BOC at the centre is a *straight angle*; and if the arc BEC is greater than a semi-circumference, as in Fig. 4, the ∠BOC at the centre is *reflex*. But the proof for Fig. 1 applies without change to both these cases, shewing that whether the given arc is greater than, equal to, or less than a semi-circumference,

the ∠BOC = *twice the* ∠BAC, *on the same arc* BEC.

DEFINITIONS

A **segment** of a circle is the figure bounded by a chord and one of the two arcs into which the chord divides the circumference.

Note. The chord of a segment is sometimes called its base.

An **angle in a segment** is one formed by two straight lines drawn from any point in the arc of the segment to the extremities of its chord.

We have seen in Theorem 32 that a circle may be drawn through *any three* points not in a straight line. But it is only under certain conditions that a circle can be drawn through more than three points.

Definition. If four or more points are so placed that a circle may be drawn through them, they are said to be **concyclic**.

THEOREM 39. [Euclid III. 21]

Angles in the same segment of a circle are equal.

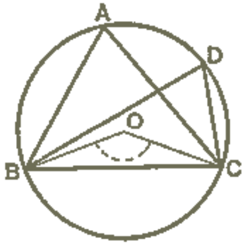


Fig. 1.

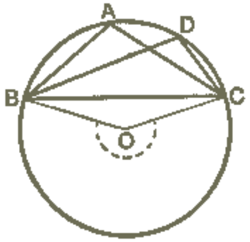


Fig. 2.

Let BAC, BDC be angles in the same segment BADC of a circle, whose centre is O.

It is required to prove that the ∠BAC = *the* ∠BDC.

Join BO, OC.

Proof. Because the ∠BOC is at the centre, and the ∠BAC at the O^ce, standing on the same arc BC,

∴ the ∠BOC = twice the ∠BAC. *Theor.* 38.

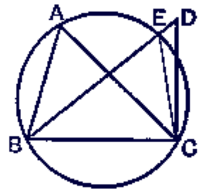
Similarly the ∠BOC = twice the ∠BDC.

∴ the ∠BAC = the ∠BDC. Q.E.D.

NOTE. The given segment may be greater than a semicircle as in Fig. 1, or less than a semicircle as in Fig. 2: in the latter case the angle BOC will be reflex. But by virtue of the extension of Theorem 38, given on the preceding page, the above proof applies equally to both figures.

CONVERSE OF THEOREM 39

Equal angles standing on the same base, and on the same side of it, have their vertices on an arc of a circle, of which the given base is the chord.



Let BAC, BDC be two equal angles standing on the same base BC, and on the same side of it.

It is required to prove that A *and* D *lie on an arc of a circle having* BC *as its chord.*

Let ABC be the circle which passes through the three points A, B, C; and suppose it cuts BD or BD produced at the point E.

Join EC.

Proof. Then the ∠BAC = the ∠BEC in the same segment.

But, by hypothesis, the ∠BAC = the ∠BDC;

∴ the ∠BEC = the ∠BDC;

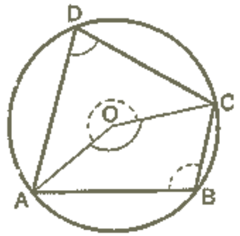
which is impossible unless E coincides with D;

∴ the circle through B, A, C must pass through D.

COROLLARY. *The locus of the vertices of triangles drawn on the same side of a given base, and with equal vertical angles, is an arc of a circle.*

Theorem 40. [Euclid III. 22]

The opposite angles of any quadrilateral inscribed in a circle are together equal to two right angles.



Let ABCD be a quadrilateral inscribed in the ⊙ABC.

It is required to proved that

(i) *the* ∠s ADC, ABC *together = two rt. angles.*

(ii) *the* ∠s BAD, BCD *together = two rt. angles.*

Suppose O is in the centre of the circle.

Join OA, OC.

Proof. Since the ∠ADC at the ○ce = half the ∠AOC at the centre, standing on the same arc ABC;
and the ∠ABC at the ○ce = half the reflex ∠AOC at the centre, standing on the same arc ADC;

∴ The ∠s ADC, ABC together = half the sum of the ∠AOC and the reflex ∠AOC.

But these angles make up four rt. angles.

∴ the ∠s ADC, ABC together = two rt. angles.

Similarly the ∠s BAD, BCD together = two rt. angles.

Q.E.D.

Note. The results of Theorems 39 and 40 should be carefully compared.

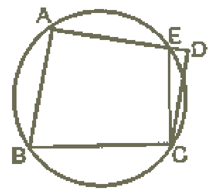
From Theorem 39 we learn that angles in the *same* segment are *equal*.

From Theorem 40 we learn that angles in *conjugate* segments are *supplementary*.

Definition. A quadrilateral is called **cyclic** when a circle can be drawn through its four vertices.

Converse of Theorem 40

If a pair of opposite angles of a quadrilateral are supplementary, its vertices are concyclic.



Let ABCD be a quadrilateral in which the opposite angles at B and D are supplementary.

It is required to prove that the points A, B, C, D *are concyclic.*

Let ABC be the circle which passes through the three points A, B, C; and suppose it cuts AD or AD produced in the point E.

Join EC.

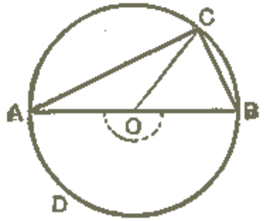
Proof. Then since ABCE is a cyclic quadrilateral,
∴ the ∠AEC is the supplement of the ∠ABC.

But, by hypothesis, the ∠ADC is the supplement of the ∠ABC
∴ the ∠AEC = the ∠ADC;
which is impossible unless E coincides with D.

∴ the circle which passes through A, B, C must pass through D:
that is, A, B, C, D are concyclic. Q.E.D.

THEOREM 41. [Euclid III. 31]

The angle in a semi-circle is a right angle.



Let ADB be a circle of which AB is a diameter and O the centre; and let C be any point on the semi-circumference ACB.

It is required to prove that the ∠ACB *is a rt. angle.*

1st Proof. The ∠ACB at the ○ᶜᵉ is half the *straight angle* AOB at the centre, standing on the same arc ADB;

and a *straight angle* = two rt. angles:

∴ the ∠ACB is a rt. angle. Q.E.D.

2nd Proof. Join OC.

Then because OA = OC,

∴ the ∠OCA = the ∠OAC. *Theor.* 5.

And because OB = OC,

∴ the ∠OCB = the ∠OBC.

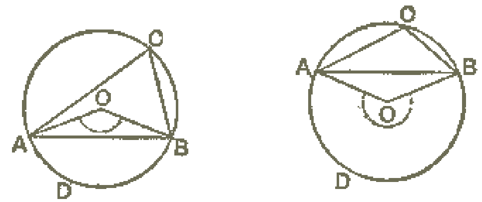
∴ the whole ∠ACB = the ∠OAC + the ∠OBC.

But the three angles of the △ACB together = two rt. angles;

∴ the ∠ACB = one-half of two rt. angles

= one rt. angle. Q.E.D.

COROLLARY. *The angle in a segment greater than a semi-circle is acute; and the angle in a segment less than a semi-circle is obtuse.*



The ∠ACB at the ○ᶜᵉ is half the ∠AOB at the centre, on the same arc ADB.

(i) If the segment ACB is greater than a semi-circle, then ADB is a *minor* arc;

∴ the ∠AOB is *less* than two rt. angles;

∴ the ∠ACB is *less* than one rt. angle.

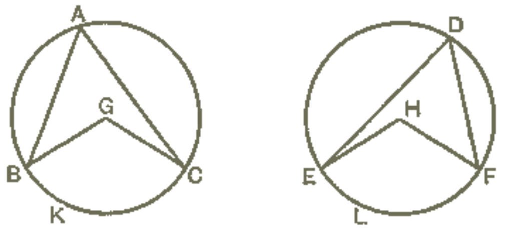
(ii) If the segment ACB is less than a semi-circle, then ADB is a *major* arc;

∴ the ∠AOB is *greater* than two rt. angles;

∴ the ∠ACB is *greater* than one rt. angle.

Theorem 42. [Euclid III. 26]

In equal circles, arcs which subtend equal values, either at the centres or at the circumferences, are equal.



Let ABC, DEF be equal circles, and let the ∠BGC = the ∠EHF at the centres; and consequently

the ∠BAC = the ∠EDF at the ○ces. *Theor.* 38.

It is required to prove that the arc BKC = *the arc* ELF.

Proof. Apply the ⊙ABC to the ⊙DEF, so that the centre G falls on the centre H, and GB falls along HE.

Then because the ∠BGC = the ∠EHF,

∴ GC will fall along HF.

And because the circles have equal radii, B will fall on E, and C on F, and the circumferences of the circles will coincide entirely.

∴ the arc BKC must coincide with the arc ELF;

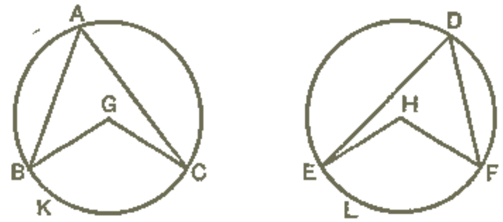
∴ the arc BKC = the arc ELF. Q.E.D.

Corollary. *In equal circles sectors which have equal angles are equal.*

Obs. It is clear that any theorem relating to arcs, angles, and chords in *equal* circles must also be true in the *same* circle.

Theorem 43. [Euclid III. 27]

In equal circles angles, either at the centres or at the circumferences, which stand on equal arcs are equal.



Let ABC, DEF be equal circles;

and let the arc BKC = the arc ELF.

It is required to prove that

the ∠BGC = *the* ∠EHF *at the centres;*

also the ∠BAC = *the* ∠EDF *at the* ○ces.

Proof. Apply the ⊙ABC to the ⊙DEF, so that the centre G falls on the centre H, and GB falls along HE.

Then because the circles have equal radii,

∴ B falls on E, and the two ○ces coincide entirely.

And, by hypothesis, the arc BKC = the arc ELF.

∴ C falls on F, and consequently GC on HF;

∴ the ∠BGC = the ∠EHF.

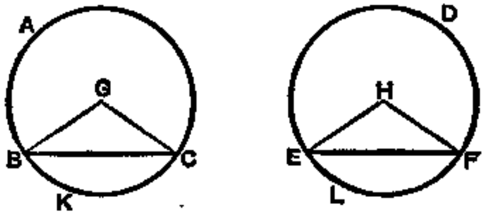
And since the ∠BAC at the ○ce = half the ∠BGC at the centre;

and likewise the ∠EDF = half the ∠EHF;

∴ the ∠BAC = the ∠EDF. Q.E.D.

THEOREM 44. [Euclid III. 28]

In equal circles, arcs which are cut off by equal chords are equal, the major arc equal to the major arc, and the minor to the minor.



Let ABC, DEF be equal circles whose centres are G and H;
and let the chord BC = the chord EF.

It is required to prove that
the major arc BAC = *the major arc* EDF,
and *the minor arc* BKC = *the minor arc* ELF.
Join BG, GC, EH, HF.

Proof. In the $\triangle^s$ BGC, EHF,

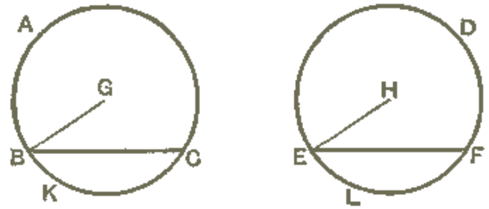
because { BG = EH, being radii of equal circles,
GC = HF, for the same reason,
and BC = EF, by hypothesis;

$\therefore$ the $\angle$ BGC = the $\angle$ EHF; *Theor.* 7.
$\therefore$ the arc BKC = the arc ELF; *Theor.* 42.
and these are the minor arcs.

But the whole $\bigcirc^{ce}$ ABKC = the whole $\bigcirc^{ce}$ DELF;
$\therefore$ the remaining arc BAC = the remaining arc EDF:
and these are the major arcs. Q.E.D.

THEOREM 45. [Euclid III. 29]

In equal circles chords which cut off equal arcs are equal.



Let ABC, DEF be equal circles whose centres are G and H;
and let the arc BKC = the arc ELF.

It is required to prove that the chord BC = *the chord* EF.
Join BG, EH.

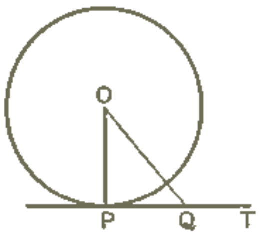
Proof. Apply the $\odot$ABC to the $\odot$DEF, so that G falls on H and GB along HE.

Then because the circles have equal radii,
$\therefore$ B falls on E, and the $\bigcirc^{ces}$ coincide entirely.

And because the arc BKC = the arc ELF,
$\therefore$ C falls on F.
$\therefore$ the chord BC coincides with the chord EF;
$\therefore$ the chord BC = the chord EF. Q.E.D.

Theorem 46

The tangent at any point of a circle is perpendicular to the radius drawn to the point of contact.



Let PT be a tangent at the point P to a circle whose centre is O.

It is required to prove that PT *is perpendicular to the radius* OP.

Proof. Take *any* point Q in PT, and join OQ.

Then since PT is a tangent, every point in it except P is outside the circle.

$\therefore$ OQ is greater than the radius OP.

And this is true for every point Q in PT;

$\therefore$ OP is the shortest distance from O to PT.

Hence OP is perp. to PT. *Theor.* 12, *Cor.* 1.

Q.E.D.

Corollary 1. Since there can be only one perpendicular to OP at the point P, it follows that *one and only one tangent can be drawn to a circle at a given point on the circumference.*

Corollary 2. Since there can be only one perpendicular to PT at the point P, it follows that *the perpendicular to a tangent at its point of contact passes through the centre.*

Corollary 3. Since there can be only one perpendicular from O to the line PT, it follows that *the radius drawn perpendicular to the tangent passes through the point of contact.*

Theorem 46. [By the Method of Limits]

The tangent at any point of a circle is perpendicular to the radius drawn to the point of contact.

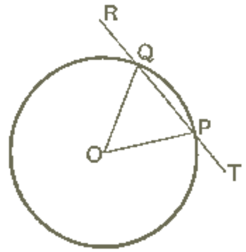


Fig. 1.

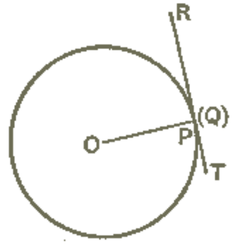


Fig. 2.

Let P be a point on a circle whose centre is O.

It is required to prove that the tangent at P *is perpendicular to the radius* OP.

Let RQPT (Fig. 1) be a secant cutting the circle at Q and P.

Join OQ, OP.

Proof. Because OP = OQ,

$\therefore$ the $\angle$OQP = the $\angle$OPQ;

$\therefore$ the supplements of these angles are equal;

that is, the $\angle$OQR = the $\angle$OPT,

and this is true however near Q is to P.

Now let the secant QP be turned about the point P so that Q continually approaches and finally coincides with P; then in the ultimate position,

(i) the secant RT becomes the tangent at P, }
(ii) OQ coincides with OP; } Fig. 2,

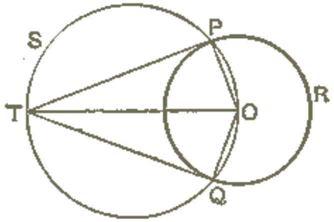
and therefore the equal $\angle$s OQR, OPT become adjacent,

$\therefore$ OP is perp. to RT. Q.E.D.

Note. The method of proof employed here is known as the **Method of Limits.**

Theorem 47

Two tangents can be drawn to a circle from an external point.



Let PQR be a circle whose centre is O, and let T be an external point.

It is required to prove that there can be two tangents drawn to the circle from T.

Join OT, and let TSO be the circle on OT as diameter.

This circle will cut the ⊙PQR in two points, since T is without, and O is within, the ⊙PQR. Let P and Q be these points.

Join TP, TQ; OP, OQ.

Proof. Now each of the ∠ˢ TPO, TQO, being in a semi-circle, is a rt. angle;

∴ TP, TQ are perp. to the radii OP, OQ respectively.

∴ TP, TQ are tangents at P and Q. *Theor.* 46.

Q.E.D.

Corollary. *The two tangents to a circle from an external point are equal, and subtend equal angles at the centre.*

For in the △ˢ TPO, TQO,

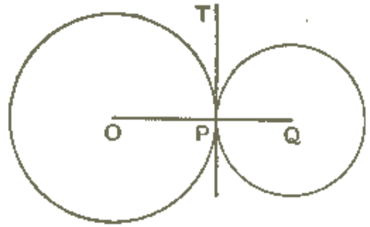
because { the ∠ˢ TPO, TQO are *right angles*
the hypotenuse TO is common,
and OP = OQ, being radii;

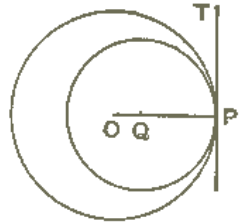
∴ TP = TQ,

and the ∠TOP = the ∠TOQ. *Theor.* 18.

Theorem 48

If two circles touch one another, the centres and the point of contact are in one straight line.





Let two circles whose centres are O and Q touch at the point P.

It is required to prove that O, P, *and* Q *are in one straight line.*

Join OP, QP.

Proof. Since the given circles touch at P, they have a common tangent at that point. *Page* 173.

Suppose PT to touch both circles at P.

Then since OP and QP are radii drawn to the point of contact,

∴ OP and QP are both perp. to PT;

∴ OP and QP are in one st. line. *Theor.* 2.

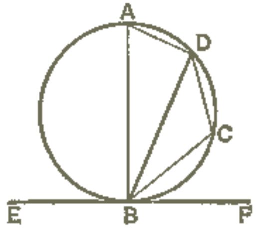
That is, the points O, P, and Q are in one st. line. Q.E.D.

Corollaries. (i) *If two circles touch externally the distance between their centres is equal to the sum of their radii.*

(ii) *If two circles touch internally the distance between their centres is equal to the difference of their radii.*

THEOREM 49. [Euclid III. 32]

The angles made by a tangent to a circle with a chord drawn from the point of contact are respectively equal to the angles in the alternate segments of the circle.



Let EF touch the ⊙ABC at B, and let BD be a chord drawn from B, the point of contact.

It is required to prove that

(i) *the* ∠FBD = *the angle in the alternate segment* BAD ;

(ii) *the* ∠EBD = *the angle in the alternate segment* BCD.

Let BA be the diameter through B, and C any point in the arc of the segment which does not contain A.

Join AD, DC, CB.

Proof. Because the ∠ADB in a semicircle is a rt. angle,

∴ the ∠s DBA, BAD together = a rt. angle.

But since EBF is a tangent, and BA a diameter,

∴ the ∠FBA is a rt. angle.

∴ the ∠FBA = the ∠s DBA, BAD together.

Take away the common ∠DBA,

then the ∠FBD = the ∠BAD, which is in the alternate segment.

Again because ABCD is a cyclic quadrilateral,

∴ the ∠BCD = the supplement of the ∠BAD

= the supplement of the ∠FBD

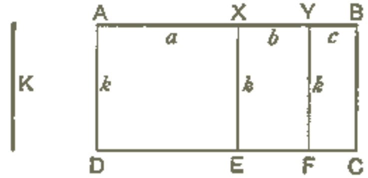
= the ∠EBD ;

∴ the ∠EBD = the ∠BCD, which is in the alternate segment.

Q.E.D.

THEOREM 50. [Euclid II. 1]

If of two straight lines, one is divided into any number of parts, the rectangle contained by the two lines is equal to the sum of the rectangles contained by the undivided line and the several parts of the divided line.



Let AB and K be the two given st. lines, and let AB be divided into any number of parts AX, XY, YB, which contain respectively a, b, and c units of length ; so that AB contains $a+b+c$ units.

Let the line K contain k units of length.

It is required to prove that

the rect. AB, K = rect. AX, K + rect. XY, K + rect. YB, K ;

namely that

$(a+b+c)k = ak + bk + ck.$

Construction. Draw AD perp. to AB and equal to K.

Through D draw DC par[l] to AB.

Through X, Y, B draw XE, YF, BC par[l] to AD.

Proof. The fig. AC = the fig. AE + the fig. XF + the fig. YC ; and of these, by construction,

fig. AC = rect. AB, K ; and contains $(a+b+c)k$ units of area ;

fig. AE = rect. AX, K ; and contains ak units of area ;
fig. XF = rect. XY, K ; .... bk .... ;
fig. YC = rect. YB, K ; .... ck ....

Hence

the rect. AB, K = rect. AX, K + rect. XY, K + rect. YB, K ;

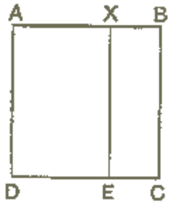
or, $(a+b+c)k = ak + bk + ck.$

Q.E.D.

*Corollaries. [Euclid II. 2 and 3]

Two special cases of this Theorem deserve attention.

(i) When AB is divided only at one point X, and when the undivided line AD is equal to AB.



Then the sq. on AB = the rect. AB, AX + the rect. AB, XB.

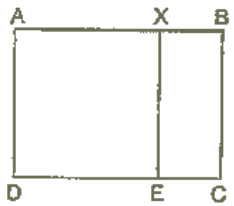
That is,

The square on the given line is equal to the sum of the rectangles contained by the whole line and each of the segments.

Or thus :

$$\begin{aligned} AB^2 &= AB \, . \, AB \\ &= AB(AX + XB) \\ &= AB \, . \, AX + AB \, . \, XB. \end{aligned}$$

(ii) When AB is divided at one point X, and when the undivided line AD is equal to one segment AX.



Then the rect. AB, AX = the sq. on AX + the rect. AX, XB.

That is,

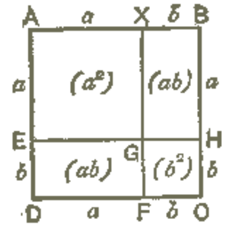
The rectangle contained by the whole line and one segment is equal to the square on that segment with the rectangle contained by the two segments.

Or thus :

$$\begin{aligned} AB \, . \, AX &= (AX + XB)AX \\ &= AX^2 + AX \, . \, XB. \end{aligned}$$

Theorem 51. [Euclid II. 4]

If a straight line is divided **internally** *at any point, the square on the given line is equal to the sum of the squares on the two segments* **together with** *twice the rectangle contained by the segments.*



Let AB be the given st. line divided *internally* at X ; and let the segments AX, XB contain a and b units of length respectively.

Then AB is the *sum* of the segments AX, XB, and therefore contains $a + b$ units.

It is required to prove that

$$AB^2 = AX^2 + XB^2 + 2AX \, . \, XB \, ;$$

namely that

$$(a+b)^2 = a^2 + b^2 + 2ab.$$

Construction. On AB describe a square ABCD. From AD cut off AE equal to AX, or a. Then ED = XB = b. Through E and X draw EH, XF par[l] respectively to AB, AD and meeting at G.

Proof. Then the fig. AC = the figs. AG, GC + the figs. EF, XH.

And of these, by construction,

fig. AC is the sq. on AB, and contains $(a+b)^2$ units of area ;

fig. AG = sq. on AX, and contains a^2 units of area ;
fig. GC = sq. on XB, ... b^2 .... ;
fig. EF = rect. EG, ED
= rect. AX, XB .... ab .... ;
fig. XH = rect. GX, XB
= rect. AX, XB .... ab ....

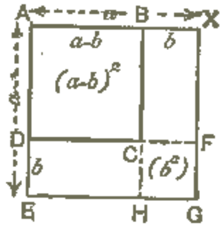
Hence $AB^2 = AX^2 + XB^2 + 2AX \, . \, XB$;

that is, $(a+b)^2 = a^2 + b^2 + 2ab.$

Q.E.D.

THEOREM 52. [Euclid II. 7]

If a straight line is divided **externally** *at any point, the square on the given line is equal to the sum of the squares on the two segments* **diminished by** *twice the rectangle contained by the segments.*



Let AB be the given st. line divided *externally* at X; and let the segments AX, XB contain a and b units of length respectively.

Then AB is the *difference* of the segments AX, XB, and therefore contains $a-b$ units.

It is required to prove that

$$AB^2 = AX^2 + XB^2 - 2AX \cdot XB;$$

namely that $(a-b)^2 = a^2 + b^2 - 2ab.$

Construction. On AX describe a square AXGE. From AE cut off AD equal to AB, or $a-b$. Then ED = XB = b. Through D and B draw DF, BH par[1] respectively to AX, AE, meeting at C.

Proof. Then the fig. AC = the figs. AG, CG − the figs. EF, XH.

And of these, by construction,

fig. AC is the sq. on AB, and contains $(a-b)^2$ units of area;

fig. AG = sq. on AX, and contains a^2 units of area;
fig. CG = sq. on XB, ... b^2 .... ;
fig. EF = rect. EG, ED
= rect. AX, XB ....ab .... ;
fig. XH = rect. GX, XB
= rect. AX, XB ....ab ....

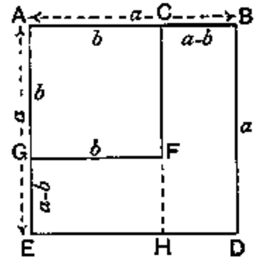
Hence $AB^2 = AX^2 + XB^2 - 2AX \cdot XB$;

that is, $(a-b)^2 = a^2 + b^2 - 2ab.$

Q.E.D.

THEOREM 53. [Euclid II. 5 and 6]

The difference of the squares on two straight lines is equal to the rectangle contained by their sum and difference.



Let the given lines AB, AC be placed in the same st. line, and let them contain a and b units of length respectively.

It is required to prove that

$$AB^2 - AC^2 = (AB + AC)(AB - AC);$$

namely that $a^2 - b^2 = (a+b)(a-b).$

Construction. On AB and AC draw the squares ABDE, ACFG; and produce CF to meet ED at H.

Then GE = CB = $a-b$ units.

Proof. Now $AB^2 - AC^2$ = the sq. AD − the sq. AF
= the rect. CD + the rect. GH
= DB . BC + GF . GE
= AB . CB + AC . CB
= (AB + AC)CB
= (AB + AC)(AB − AC).

That is,

$$a^2 - b^2 = (a+b)(a-b).$$

Q.E.D.

COROLLARY. *If a straight line is bisected, and also divided (internally or externally) into two unequal segments, the rectangle contained by these segments is equal to the difference of the squares on half the line and on the line between the points of section.*

Fig. 1.

Fig. 2.

That is, if AB is bisected at X and also divided at Y, internally in Fig. 1, and externally in Fig. 2, then

in Fig. 1, $AY \cdot YB = AX^2 - XY^2$;

in Fig. 2, $AY \cdot YB = XY^2 - AX^2$.

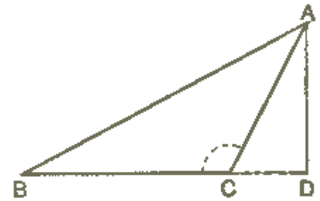
For in the first case,

$$AY \cdot YB = (AX + XY)(XB - XY)$$
$$= (AX + XY)(AX - XY)$$
$$= AX^2 - XY^2.$$

The second case may be similarly proved.

THEOREM 54. [Euclid II. 12]

In an obtuse-angled triangle, the square on the side subtending the obtuse angle is equal to the sum of the squares on the sides containing the obtuse angle **together with** *twice the rectangle contained by one of those sides and the* **projection of the other side** *upon it.*



Let ABC be a triangle obtuse-angled at C; and let AD be drawn perp. to BC produced, so that CD is the projection of the side CA on BC. [See Def. p. 63.]

It is required to prove that

$$AB^2 = BC^2 + CA^2 + 2BC \cdot CD.$$

Proof. Because BD is the *sum* of the lines BC, CD,

$\therefore$ $BD^2 = BC^2 + CD^2 + 2BC \cdot CD$. *Theor.* 51.

To each of these equals add DA^2.

Then $BD^2 + DA^2 = BC^2 + (CD^2 + DA^2) + 2BC \cdot CD$.

But $BD^2 + DA^2 = AB^2$ and $CD^2 + DA^2 = CA^2$, for the $\angle D$ is a rt. $\angle$.

Hence $AB^2 = BC^2 + CA^2 + 2BC \cdot CD$.

Q.E.D.

THEOREM 55. [Euclid II. 13]

In every triangle the square on the side subtending an acute angle is equal to the sum of the squares on the sides containing that angle **diminished by** *twice the rectangle contained by one of those sides and the* **projection of the other side** *upon it.*

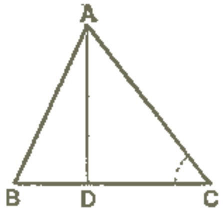


Fig. 1.

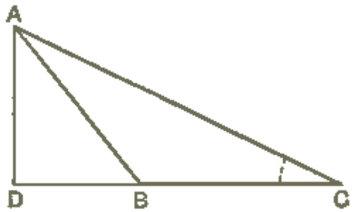


Fig. 2.

Let ABC be a triangle in which the $\angle C$ is acute; and let AD be drawn perp. to BC, or BC produced; so that CD is the projection of the side CA on BC.

It is required to prove that

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$$

Proof. Since in both figures BD is the *difference* of the lines BC, CD,

$$\therefore \; BD^2 = BC^2 + CD^2 - 2BC \cdot CD. \qquad \textit{Theor. } 52.$$

To each of these equals add DA^2.

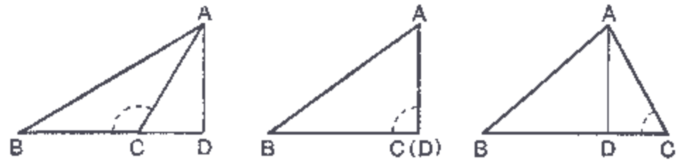
Then $BD^2 + DA^2 = BC^2 + (CD^2 + DA^2) - 2BC \cdot CD.$ ......(i)

But $BD^2 + DA^2 = AB^2$ and $CD^2 + DA^2 = CA^2$, for the $\angle D$ is a rt. $\angle$.

Hence $AB^2 = BC^2 + CA^2 - 2BC \cdot CD.$

Q.E.D.

SUMMARY OF THEOREMS 29, 54 and 55



(i) If the $\angle ACB$ is *obtuse*,

$$AB^2 = BC^2 + CA^2 + 2BC \cdot CD. \qquad \textit{Theor. } 54.$$

(ii) If the $\angle ACB$ is *a right angle*,

$$AB^2 = BC^2 + CA^2. \qquad \textit{Theor. } 29.$$

(iii) If the $\angle ACB$ is *acute*,

$$AB^2 = BC^2 + CA^2 - 2BC \cdot CD. \qquad \textit{Theor. } 55.$$

Observe that in (ii), when the $\angle ACB$ is *right*, AD coincides with AC, so that CD (the projection of CA) vanishes;

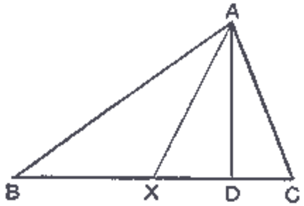
hence, in this case, $2BC \cdot CD = 0$.

Thus the three results may be collected in a single enunciation:

The square on a side of a triangle is greater than, equal to, or less than the sum of the squares on the other sides, according as the angle contained by those sides is obtuse, a right angle, or acute; the difference in cases of inequality being twice the rectangle contained by one of the two sides and the projection on it of the other.

THEOREM 56

In any triangle the sum of the squares on two sides is equal to twice the square on half the third side together with twice the square on the median which bisects the third side.



Let ABC be a triangle, and AX the median which bisects the base BC.

It is required to prove that

$$AB^2 + AC^2 = 2BX^2 + 2AX^2.$$

Draw AD perp. to BC; and consider the case in which AB and AC are unequal, and AD falls within the triangle.

Then of the $\angle^s$ AXB, AXC, one is obtuse, and the other acute. Let the $\angle$ AXB be obtuse.

Then from the $\triangle$ AXB,

$$AB^2 = BX^2 + AX^2 + 2BX \cdot XD. \qquad \textit{Theor. } 54.$$

And from the $\triangle$ AXC,

$$AC^2 = XC^2 + AX^2 - 2XC \cdot XD. \qquad \textit{Theor. } 55.$$

Adding these results, and remembering that $XC = BX$, we have

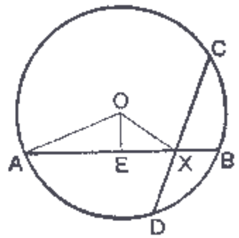
$$AB^2 + AC^2 = 2BX^2 + 2AX^2.$$

Q.E.D.

NOTE.—The proof may easily be adapted to the case in which the perpendicular AD falls outside the triangle.

THEOREM 57. [Euclid III. 35]

If two chords of a circle cut at a point within it, the rectangles contained by their segments are equal.



In the ⊙ABC, let AB, CD be chords cutting at the internal point X.

It is required to prove that

the rect. AX, XB = *the rect.* CX, XD.

Let O be the centre, and r the radius, of the given circle.

Supposing OE drawn perp. to the chord AB, and therefore bisecting it.

Join OA, OX.

Proof. The rect. AX, XB $= (AE + EX)(EB - EX)$

$$= (AE + EX)(AE - EX)$$

$$= AE^2 - EX^2 \qquad \textit{Theor. } 53.$$

$$= (AE^2 + OE^2) - (EX^2 + OE^2)$$

$$= \quad r^2 \quad - \quad OX^2,$$

since the $\angle^s$ at E are rt. $\angle^s$.

Similarly it may be shewn that

the rect. CX, XD $= r^2 - OX^2$.

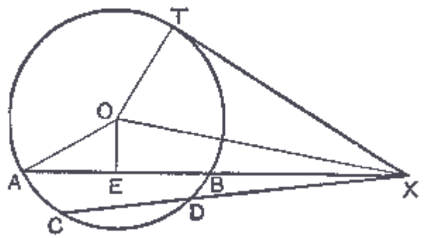
$\therefore$ the rect. AX, XB = the rect. CX, XD.

Q.E.D.

COROLLARY. *Each rectangle is equal to the square on half the chord which is bisected at the given point* X.

THEOREM 58. [Euclid III. 36]

If two chords of a circle, when produced, cut at a point outside it, the rectangles contained by their segments are equal. And each rectangle is equal to the square on the tangent from the point of intersection.



In the ⊙ABC, let AB, CD be chords cutting, when produced, at the external point X; and let XT be a tangent drawn from that point.

It is required to prove that

the rect. AX, XB = *the rect.* CX, XD = *the sq. on* XT.

Let O be the centre, and r the radius of the given circle.

Suppose OE drawn perp. to the chord AB, and therefore bisecting it.

Join OA, OX, OT.

Proof. The rect. AX, XB $= (EX + AE)(EX - EB)$

$= (EX + AE)(EX - AE)$

$= EX^2 - AE^2$ *Theor.* 53.

$= (EX^2 + OE^2) - (AE^2 + OE^2)$

$= OX^2 - r^2$, since the $\angle^s$ at E are rt. $\angle^s$.

Similarly it may be shewn that

the rect. CX, XD $= OX^2 - r^2$.

And since the radius OT is perp. to the tangent XT,

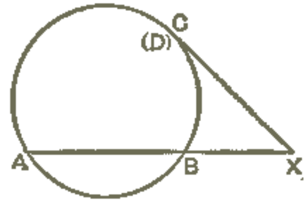
$\therefore XT^2 = OX^2 - r^2$. *Theor.* 29.

$\therefore$ the rect. AX, XB = the rect. CX, XD = the sq. on XT.

Q.E.D.

THEOREM 59. [Euclid III. 37]

If from a point outside a circle two straight lines are drawn, one of which cuts the circle, and the other meets it; and if the rectangle contained by the whole line which cuts the circle and the part of it outside the circle is equal to the square on the line which meets the circle, then the line which meets the circle is a tangent to it.



From X a point outside the ⊙ABC, let two straight lines XA, XC be drawn, of which XA cuts the circle at A and B, and XC meets it at C;

and let the rect. XA . XB = the sq. on XC.

It is required to prove that XC *touches the circle at* C.

Proof. Suppose XC meets the circle again at D;

then XA . XB = XC . XD. *Theor.* 58.

But by hypothesis, $XA \cdot XB = XC^2$;

$\therefore XC \cdot XD = XC^2$;

$\therefore XD = XC$.

Hence XC cannot meet the circle again unless the points of section coincide;

that is, XC is a tangent to the circle.

Q.E.D.

NOTE ON THEOREMS 57, 58

Remembering that the segments into which the chord AB is divided at X, internally in Theorem 57, and externally in Theorem 58, are in each case AX, XB, we may include both Theorems in a single enunciation.

If any number of chords of a circle are drawn through a given point within or without a circle, the rectangles contained by the segments of the chords are equal.

THEOREM 60. [Euclid VI. 2]

A straight line drawn parallel to one side of a triangle cuts the other two sides, or those sides produced, proportionally.

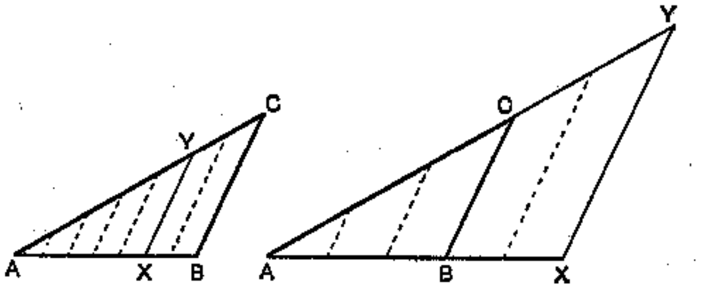


Fig. 1 Fig. 2

In the $\triangle$ABC, let XY, drawn par[1] to the side BC, cut AB, AC at X and Y, internally in Fig. 1, externally in Fig. 2.

It is required to prove in both cases that

$$AX : XB = AY : YC.$$

Proof. Suppose X divides AB in the ratio $m : n$; that is, suppose $AX : XB = m : n$;

so that, if AX is divided into m equal parts, then XB may be divided into n such equal parts.

Through the points of division in AX, XB let parallels be drawn to BC.

Then these parallels divide the segments AY, YC into parts which are all equal; *Theor.* 22.

and of these equal parts AY contains m,

and YC contains n;

hence $AY : YC = m : n$.

$$\therefore AX : XB = AY : YC.$$

Q.E.D.

Conversely, *if a line cuts two sides of a triangle proportionally, it is parallel to the third side.*

Fig. 1. Fig. 2.

Conversely, let XY cut the sides AB, AC proportionally, so that

$$AX : XB = AY : YC.$$

It is required to prove that XY *is parallel to* BC.

Let XP be drawn through X par[1] to BC, to meet AC in P.

Then $AP : PC = AX : XB$;

but, by hypothesis, $AY : YC = AX : XB$.

Thus AC is cut, internally in Fig. 1, and externally in Fig. 2 in the same ratio at P and Y.

Hence P coincides with Y, and consequently XP with XY. *Theor.* VI. p. 252.

That is, XY is par[1] to BC.

Q.E.D.

COROLLARY. If XY is parallel to BC, then

$$AX : AB = AY : AC.$$

For, taking Fig. 1, it may be shewn that

$$AX : AB = m : m + n;$$

and hence, by Theorem 22, that

$$AY : AC = m : m + n.$$

$$\therefore AX : AB = AY : AC.$$

Conversely, if $AX : AB = AY : AC$,

it may be proved as above that XY is par[1] to BC.

Theorem 61. [Euclid VI. 3 and A]

If the vertical angle of a triangle is bisected internally or externally, the bisector divides the base internally or externally into segments which have the same ratio as the other sides of the triangle.

Conversely, if the base is divided internally or externally into segments proportional to the other sides of the triangle, the line joining the point of section to the vertex bisects the vertical angle internally or externally.

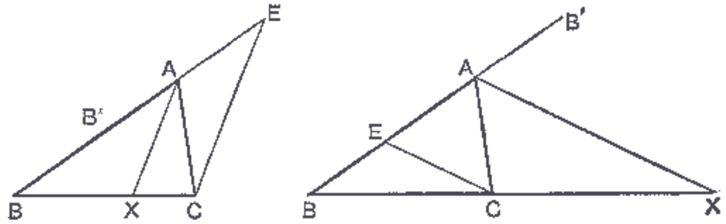


Fig. 1. Fig. 2.

In the $\triangle ABC$, let AX bisect the $\angle BAC$, internally in Fig. 1, and externally in Fig. 2; that is, in the latter case, let AX bisect the exterior $\angle B'AC$.

It is required to prove in both cases that

$$BX : XC = BA : AC.$$

Let CE be drawn through C par[l] to XA to meet BA (produced, if necessary) at E. In Fig. 1 let a point B′ be taken in AB.

Proof. Because XA and CE are par[l],

$\therefore$, in both Figs., the $\angle B'AX =$ the int. opp. $\angle AEC$.

Also, by hypothesis,

$$\text{the } \angle B'AX = \text{the } \angle XAC$$
$$= \text{the alt. } \angle ACE.$$
$$\therefore \text{ the } \angle AEC = \text{the } \angle ACE:$$
$$\therefore AC = AE.$$

Again, because XA is par[l] to CE, a side of the $\triangle BCE$,

$\therefore$, in both Figs., $BX : XC = BA : AE$;

that is, $BX : XC = BA : AC$.

Q.E.D.

Conversely, let BC be divided internally (Fig. 1) or externally (Fig. 2) at X, so that $BX : XC = BA : AC$.

It is required to prove that the $\angle B'AX =$ the $\angle XAC$.

Proof. For, with the same construction as before,

because XA is par[l] to CE, a side of the $\triangle BCE$,

$$\therefore BX : XC = BA : AE.$$

But, by hypothesis, $BX : XC = BA : AC$;

$$\therefore BA : AC = BA : AE;$$
$$\therefore AC = AE.$$
$$\therefore \text{ the } \angle AEC = \text{the } \angle ACE$$
$$= \text{the alt. } \angle XAC$$

And in both Figs.,

$$\text{the ext. } \angle B'AX = \text{the int. opp. } \angle AEC;$$
$$\therefore \text{ the } \angle B'AX = \text{the } \angle XAC.$$

Q.E.D.

Definition

When a finite straight line is divided internally and externally into segments which have the same ratio, it is said to **be cut harmonically.**

Hence the following Corollary to Theorem 61.

The base of a triangle is divided harmonically by the internal and external bisectors of the vertical angle:

for in each case the segments of the base are in the ratio of the other sides of the triangle

Theorem 62. [Euclid VI. 4]

If two triangles are equiangular to one another, their corresponding sides are proportional.

Let the $\triangle^s$ ABC, DEF have the $\angle^s$ A and B respectively equal to the $\angle^s$ D and E; and consequently the $\angle$ C equal to the $\angle$ F.

It is required to prove that

AB : DE = BC : EF = CA : FD.

Proof. Apply the $\triangle$ DEF to the $\triangle$ ABC, so that E falls on B, and EF along BC;

then since the $\angle$ E = the $\angle$ B, ED will fall along BA.

Let G and H be the points at which D and F fall respectively; so that GBH represents the $\triangle$ DEF in its new position.

Now, by hypothesis, the $\angle$ D = the $\angle$ A;

that is, the ext. $\angle$ BGH = the int. opp. $\angle$ BAC;

$\therefore$ GH is par^l to AC.

Hence BA : BG = BC : BH; *Theor.* 60, *Cor.*

that is, AB : DE = BC : EF.

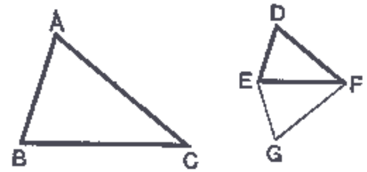
Similarly, by applying the $\triangle$ DEF to the $\triangle$ ABC, so that F falls on C, and FE, FD along CB, CA, it may be shewn that

BC : EF = CA : FD.

Hence finally, AB : DE = BC : EF = CA : FD. Q.E.D.

Theorem 63. [Euclid VI. 5]

If two triangles have their sides proportional when taken in order, the triangles are equiangular to one another, and those angles are equal which are opposite to corresponding sides.



In the $\triangle^s$ ABC, DEF, let

AB : DE = BC : EF = CA : FD.

It is required to prove that the $\triangle^s$ABC, DEF *are equiangular to one another.*

At E in FE make the $\angle$ FEG equal to the $\angle$ B;

and at F in EF make the $\angle$ EFG equal to the $\angle$ C.

$\therefore$ the remaining $\angle$ EGF = the remaining $\angle$ A.

Proof. Since the $\triangle^s$ ABC, GEF are equiangular to one another,

$\therefore$ AB : GE = BC : EF. *Theor.* 62.

But, by hypothesis, AB : DE = BC : EF;

$\therefore$ AB : GE = AB : DE.

$\therefore$ GE = DE.

Similarly GF = DF.

Then in the $\triangle^s$ GEF, DEF,

because { GE = DE,
GF = DF,
and EF is common;

$\therefore$ the triangles are identically equal; *Theor.* 7.

$\therefore$ the $\angle$ DEF = the $\angle$ GEF

= the $\angle$ B;

and the $\angle$ DFE = the $\angle$ GFE

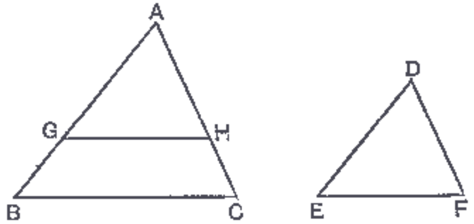
= the $\angle$ C.

$\therefore$ the remaining $\angle$ D = the remaining $\angle$ A;

that is, the $\triangle$ DEF is equiangular to the $\triangle$ ABC. Q.E.D.

THEOREM 64. [Euclid VI. 6]

If two triangles have one angle of the one equal to one angle of the other, and the sides about the equal angles proportionals, the triangles are similar.



In the $\triangle^s$ ABC, DEF, let the ∠A = the ∠D,

and let AB : DE = AC : DF.

It is required to prove that the $\triangle^s$ ABC, DEF *are similar.*

Proof. Apply the △DEF to the △ABC, so that D falls on A, and DE along AB;

then because the ∠EDF = the ∠BAC, DF must fall along AC.

Let G and H be the points at which E and F fall respectively; so that AGH represents the △DEF in its new position.

Now, by hypothesis, AB : DE = AC : DF;

that is, AB : AG = AC : AH;

hence GH is par[1] to BC. *Theor.* 60, *Cor.*

∴ the ext. ∠AGH, namely the ∠E, = the int. opp. ∠ABC;

and the ext. ∠AHG, namely the ∠F, = the int. opp. ∠ACB.

Hence the $\triangle^s$ ABC, DEF are equiangular to one another,

so that their corresponding sides are proportional; *Theor.* 62.

that is, the $\triangle^s$ ABC, DEF are similar.

Q.E.D.

* THEOREM 65. [Euclid VI. 7]

If two triangles have one angle of the one equal to one angle of the other, and the sides about another angle in one proportional to the corresponding sides of the other, then the third angles are either equal or supplementary; and in the former case the triangles are similar.

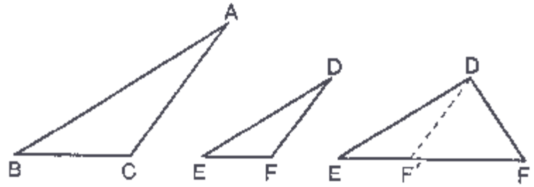


Fig. 1. Fig. 2. Fig. 3.

In the $\triangle^s$ ABC, DEF, let the ∠B = the ∠E; and let the sides about the $\angle^s$ A and D be proportional,

namely AB : DE = AC : DF.

It is required to prove that

either the ∠C = the ∠F, [as in Figs. 1 and 2];

or the ∠C = *the supplement of the* ∠F. [Figs. 1 and 3.]

Proof. (i) If the ∠A = the ∠D, [Figs. 1 and 2],

then the ∠C = the ∠F; *Theor.* 16.

and the $\triangle^s$ are equiangular, and therefore similar.

(ii) But if the ∠A is not equal to the ∠EDF [Figs. 1 and 3]

let the ∠EDF′ = the ∠A.

Then the $\triangle^s$ ABC, DEF′ are equiangular to one another;

∴ AB : DE = AC : DF′.

But by hypothesis, AB : DE = AC : DF;

∴ AC : DF′ = AC : DF.

∴ DF′ = DF.

∴ the ∠DFF′ = the ∠DF′F.

But the ∠C = the ∠DF′E *Proved.*

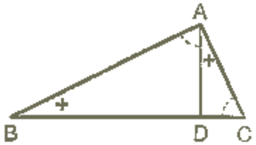
= the supplement of the ∠DF′F

= the supplement of the ∠DFE.

Q.E.D.

THEOREM 66. [Euclid VI. 8]

In a right-angled triangle, if a perpendicular is drawn from the right angle to the hypotenuse, the triangles on each side of it are similar to the whole triangle and to one another.



Let BAC be a triangle right-angled at A, and let AD be drawn perp. to BC.

It is required to prove that the Δ^s BDA, ADC *are similar to the* ΔBAC *and to one another.*

In the Δ^s BDA, BAC,

the $\angle$BDA = the $\angle$BAC, being rt. angles,

and the $\angle$B is common to both;

$\therefore$ the remaining $\angle$BAD = the remaining $\angle$BCA; *Theor.* 16.

hence the ΔBCA is equiangular to the ΔBAC;

$\therefore$ their corresponding sides are proportional;

$\therefore$ the Δ^s BDA, BAC are similar.

In the same way it may be proved that the Δ^s ADC, BAC are similar.

Hence the Δ^s BDA, ADC, having their angles severally equal to those of the ΔBAC, are equiangular to one another;

$\therefore$ they are similar.

Q.E.D.

COROLLARY. (i) Because the Δ^s DBA, DAC are similar,

$\therefore$ DB : DA = DA : DC;

that is, DA is a mean proportional between DB and DC:

and $DA^2 = DB \cdot DC$.

(ii) Because the Δ^s BCA, BAD are similar,

$\therefore$ BC : BA = BA : BD;

hence $BA^2 = BC \cdot BD$.

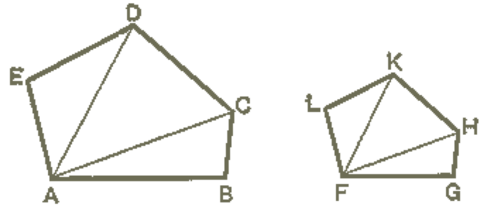
(iii) Because the Δ^s CBA, CAD are similar,

$\therefore$ CB : CA = CA : CD;

hence $CA^2 = CB \cdot CD$.

THEOREM 67

Similar polygons can be divided into the same number of similar triangles; and the lines joining corresponding vertices in each figure are proportional.



Let ABCDE, FGHKL be similar polygons, the vertex A corresponding to the vertex F, B to G, and so on. Let AC, AD be joined, and also FH, FK.

It is required to prove that

(i) the Δ^s ABC, FGH *are similar; as also the* Δ^s ACD, FHK, *and the* Δ^s ADE, FKL.

(ii) AB : FG = AC : FH = AD : FK.

Proof. (i) Since the polygons are similar,

the $\angle$ABC = the $\angle$FGH,

and AB : FG = BC : GH;

$\therefore$ the Δ^s ABC, FGH are similar. *Theor.* 64.

$\therefore$ the $\angle$BCA = the $\angle$GHF;

but because the polygons are similar,

the $\angle$BCD = the $\angle$GHK;

$\therefore$ the $\angle$ACD = the $\angle$FHK.

Also AC : FH = BC : GH, for the Δ^s ABC, FGH are similar,

= CD : HK, for the polygons are similar.

That is, the sides about the equal $\angle^s$ ACD, FHK are proportional,

$\therefore$ the Δ^s ACD, FHK are similar. *Theor.* 64.

In the same way the Δ^s ADE, FKL are similar.

(ii) And AB : FG = AC : FH, from the similar Δ^s ABC, FGH;

= AD : FK, from the similar Δ^s CAD, HFK.

Q.E.D.

NOTE. In the last Theorem the polygons have been divided into similar triangles by lines drawn from a pair of *corresponding vertices*. But this restriction is not necessary.

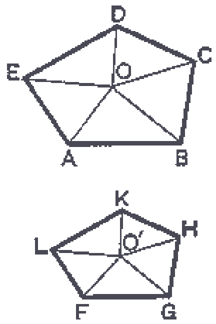
For take *any* point O in the polygon ABCDE, and join it to each of the vertices.

In the polygon FGHKL, make the ∠GFO′ equal to the ∠BAO,

and make the ∠FGO′ equal to the ∠ABO.

Join O′ to each vertex of the polygon FGHKL.

We leave as an exercise to the student the proof that the two polygons are thus divided into the same number of similar triangles.



THEOREM 68

Any two similar rectilineal figures may be so placed that the lines joining corresponding vertices are concurrent.

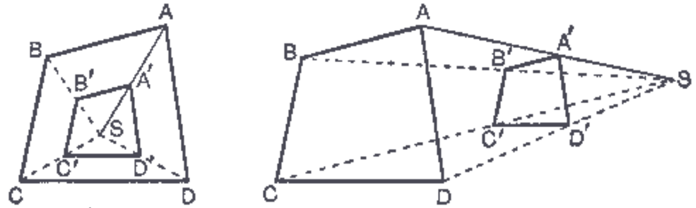


Fig. 1 Fig. 2

Let ABCD, A′B′C′D′ be similar figures.

Then since the ∠B′ = the ∠B, the figures can be so placed that A′B′, B′C′ are respectively par^l to AB, BC. It follows, since the figures are equiangular to one another, that C′D′ is par^l to CD, and D′A′ par^l to DA.

It is required to prove that when corresponding sides of the given figures are parallel, then AA′, BB′, CC′, DD′ *are concurrent.*

Join AA′, and divide it externally at S in the ratio AB : A′B′.

Join SB *and* SB′ : it will be shewn that SB and SB′ are in one straight line.

Proof. In the △^s SAB, SA′B′, since AB and A′B′ are par^l,

∴ the ∠SAB = the ∠SA′B′ ;

and, by construction, SA : SA′ = AB : A′B′ ;

∴ the △^s SAB, SA′B′ are equiangular to one another; *Theor.* 64.

∴ the ∠ASB = the ∠A′SB′.

Hence SB, SB′ are in the same st. line ;

that is, BB′ passes through the fixed point S.

Similarly CC′ and DD′ may be shewn to pass through S.

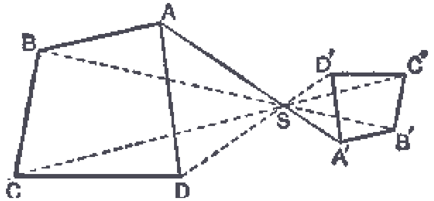
That is, AA′, BB′, CC′, DD′ are concurrent. Q.E.D.

NOTE. Observe that the joining lines AA′, BB′, CC′, DD′ are all divided externally at S in the ratio of any pair of corresponding sides of the given figures.

NOTE. In placing the given figures so that A′B′, B′C′ are respectively parallel to AB, BC, two cases arise :

(i) A′B′ and AB may have *the same sense*, as in Figs. 1 and 2 ;

(ii) A′B′ and AB..............*opposite senses*, as in the Fig. below.

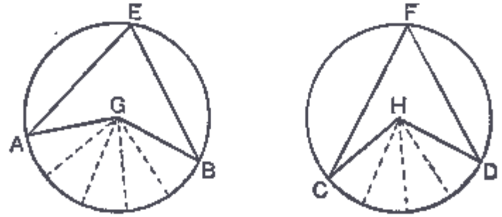


In the latter case it follows also that C′D′ is par[l] to CD, and D′A′ par[l] to DA, and it may be proved as before that AA′, BB′, CC′, DD′ are concurrent; but here S divides AA′ *internally* in the ratio of corresponding sides, and the position of the figures is *transverse*.

In each case the point S is called a **centre of similarity, or homothetic centre**; and similar figures so placed are said to be **homothetic**.

THEOREM 69. [Euclid VI. 33]

In equal circles, angles, whether at the centres or circumferences, have the same ratio as the arcs on which they stand.



Let ABE, CDF be equal circles; and let the ∠s AGB, CHD at the centres, and the ∠s AEB, CFD at the ○ces, stand on the arcs AB, CD.

It is required to prove that

(i) *the* ∠AGB : *the* ∠CHD = *the arc* AB : *the arc* CD ;

(ii) *the* ∠AEB : *the* ∠CFD = *the arc* AB : *the arc* CD.

Proof. Suppose the arc AB : the arc CD = $m : n$;
so that, if the arc AB is divided into m equal parts, then the arc CD may be divided into n such equal parts.

In each circle let radii be drawn to the points of division of the arcs AB, CD.

Then the ∠s AGB, CHD, in equal circles, are divided into angles which stand on equal arcs, and are therefore all equal.

And of these equal angles the ∠AGB contains m,
and the ∠CHD contains n ;

∴ the ∠AGB : the ∠CHD = $m : n$.

Hence the ∠AGB : the ∠CHD = the arc AB : the arc CD.

And since the ∠AEB = one half of the ∠AGB ; *Theor.* 38.
and the ∠CFD = one half of the ∠CHD ;

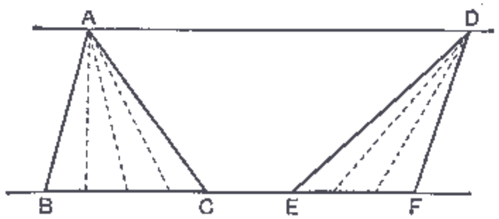
∴ the ∠AEB : the ∠CFD = the arc AB : the arc CD.

Q.E.D.

COROLLARY. Since *in equal circles, sectors which have equal angles are equal* [Theor. 42, Cor.], it may be proved as above that *the sector* AGB : *the sector* CHD = *the arc* AB : *the arc* CD.

THEOREM 70. [Euclid VI. 1]

The areas of triangles of equal altitude are to one another as their bases.



Let ABC, DEF be two triangles of equal altitude, standing on the bases BC, EF.

It is required to prove that

the △ABC : *the* △DEF = BC : EF.

Proof. Let the triangles be placed so that the bases BC, EF are in the same st. line, and the triangles on the same side of the line.

Join AD; then AD is par[l] to BF. *Def.* 2. p. 99.

Suppose the base BC : the base EF $= m : n$;
so that, if BC is divided into m equal parts, then EF may be divided into n such equal parts.

In each triangle let st. lines be drawn from the vertex to the points of division in the bases BC, EF.

Then the △[s] ABC, DEF are divided into triangles which stand on equal bases, and have the same altitude, and are therefore all equal.

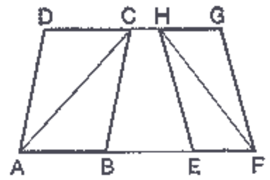
And of these equal △[s], the △ABC contains m;
and the △DEF contains n.

∴ the △ABC : the △DEF $= m : n$.

Hence the △ABC : the △DEF = BC : EF.

Q.E.D.

COROLLARY. *The areas of parallelograms of equal altitude are to one another as their bases.*



For let DB, EG be par[ms] of the same altitude, standing on the bases AB, EF.

Join AC, HF.

Then
since the par[m] DB = twice the △CAB;
and the par[m] EG = twice the △HEF:
∴ the par[m] DB : the par[m] EG = the △CAB : the △HEF
= AB : EF.

ALTERNATIVE PROOF OF THEOREM 70

Let p represent the altitude of each of the △[s] ABC, DEF.

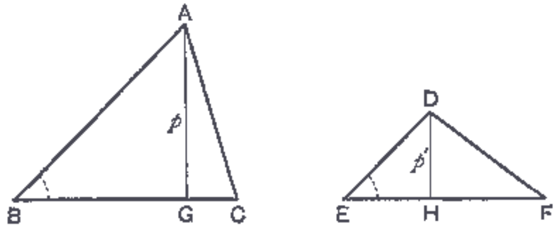
Then the area of the △ABC $= \frac{1}{2}$. base × altitude $= \frac{1}{2}$. BC $\times p$;

and the area of the △DEF = ... $= \frac{1}{2}$. EF $\times p$.

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{\frac{1}{2}.BC \times p}{\frac{1}{2}.EF \times p} = \frac{BC}{EF}.$$

THEOREM 71

If two triangles have one angle of the one equal to one angle of the other, their areas are proportional to the rectangles contained by the sides about the equal angles.



In the $\triangle^s$ ABC, DEF, let the $\angle^s$ at B and E be equal.

It is required to prove that

the $\triangle$ABC : the $\triangle$DEF = AB . BC : DE . EF.

Let AG and DH be drawn perp. to BC, EF respectively, and denote the lengths of these perp^s by p and p'.

Proof. The $\triangle ABC = \frac{1}{2}BC.p$; and the $\triangle DEF = \frac{1}{2}EF.p'$.

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BC.p}{EF.p'}. \quad \text{......................(i)}$$

But since the $\angle$B = the $\angle$E, and the $\angle$G = the $\angle$H,

$\therefore$ the $\triangle^s$ ABG, DEH are equiangular to one another; *Theor.* 16.

$$\therefore \frac{p}{p'} = \frac{AB}{DE}. \quad \text{...........(ii)} \quad \textit{Theor. } 62.$$

Substituting for $\frac{p}{p'}$ in (i),

$$\frac{\triangle ABC}{\triangle DEF} = \frac{BC.AB}{EF.DE};$$

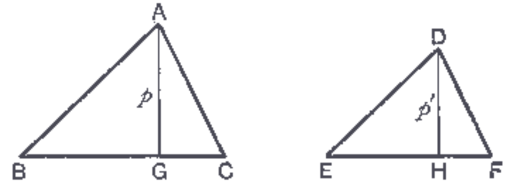
or the $\triangle$ABC : the $\triangle$DEF = AB . BC : DE . EF.

Q.E.D.

COROLLARY. *Similarly it may be shewn that* parallelograms *having one angle of the one equal to one angle of the other are proportional to the rectangles contained by the sides about the equal angles.*

THEOREM 72. [Euclid VI. 19]

The areas of similar triangles are proportional to the squares on corresponding sides.



Let ABC, DEF be similar triangles, in which BC and EF are corresponding sides.

It is required to prove that

the $\triangle$ABC : the $\triangle$DEF = BC^2 : EF^2.

Let AG and DH be drawn perp. to BC, EF respectively; and denote these perp^s by p and p'.

Proof. The $\triangle ABC = \frac{1}{2}BC.p$; and the $\triangle DEF = \frac{1}{2}EF.p'$.

$$\therefore \frac{\triangle ABC}{\triangle DEF} = \frac{BC.p}{EF.p'} \quad \text{........................(i)}$$

But since the $\angle$B = the $\angle$E, from the similar $\triangle^s$ ABC, DEF, and the $\angle$G = the $\angle$H, being right angles;

$\therefore$ the $\triangle^s$ ABG, DEH are equiangular to one another; *Theor.* 16.

$$\therefore \frac{p}{p'} = \frac{AB}{DE} \qquad \textit{Theor. } 62.$$

$$= \frac{BC}{EF}, \text{ from the similar } \triangle^s \text{ ABC, DEF.}$$

Substituting for $\frac{p}{p'}$ in (i),

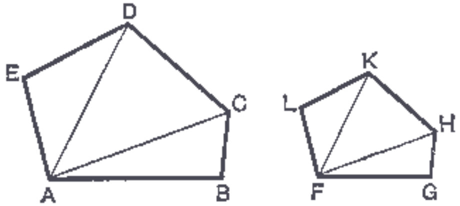
$$\frac{\triangle ABC}{\triangle DEF} = \frac{BC.BC}{EF.EF} = \frac{BC^2}{EF^2};$$

or, the $\triangle$ABC : the $\triangle$DEF = BC^2 : EF^2.

Q.E.D.

THEOREM 73. [Euclid VI. 20]

The areas of similar polygons are proportional to the squares on corresponding sides.



Let ABCDE, FGHKL be similar polygons, and let AB, FG be corresponding sides.

It is required to prove that

the polygon ABCDE : *the polygon* FGHKL $=$ AB^2 : FG^2.

Join AC, AD, FH, FK.

Proof. Then the $\triangle$s ABC, FGH are similar; *Theor.* 67.

also the $\triangle$s ACD, FHK are similar;

and the $\triangle$s ADE, FKL are similar.

$\therefore$ the $\triangle$ABC : the $\triangle$FGH $=$ AC^2 : FH^2 *Theor.* 72.

$=$ the $\triangle$ACD : the $\triangle$FHK.

Similarly,

the $\triangle$ACD : the $\triangle$FHK $=$ AD^2 : FK^2

$=$ the $\triangle$ADE : the $\triangle$FKL.

Hence $$\frac{\triangle ABC}{\triangle FGH}=\frac{\triangle ACD}{\triangle FHK}=\frac{\triangle ADE}{\triangle FKL}.$$

And in this series of equal ratios, the sum of the antecedents is to the sum of the consequents as each antecedent is to its consequent; *Theor.* V. p. 251.

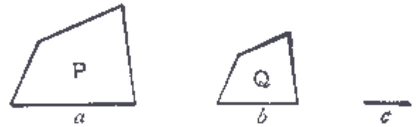
$\therefore$ the fig. ABCDE : the fig. FGHKL $=$ the $\triangle$ABC : the $\triangle$FGH

$=$ AB^2 : FG^2.

Q.E.D.

COROLLARY 1. Let a, b, c represent three lines in proportion, so that

$$\frac{a}{b}=\frac{b}{c}; \text{ and consequently } b^2=ac.$$



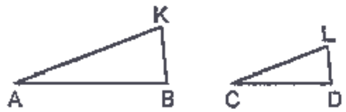
Now suppose similar figures P and Q to be drawn on a and b as corresponding sides,

then $$\frac{\text{Fig. P}}{\text{Fig. Q}}=\frac{a^2}{b^2}=\frac{a^2}{ac}=\frac{a}{c}.$$

Hence if three straight lines are proportionals, and any similar figures are drawn on the first and second as corresponding sides, then the fig. on the **first** : *the fig. on the* **second** $=$ *the* **first** : *the* **third.**

COROLLARY 2. Let

AB : CD $=$ EF : GH;



and let similar figures KAB, LCD be similarly described on AB, CD, and also let similar figures MF, NH be similarly described on EF, GH.

Then since $$\frac{AB}{CD}=\frac{EF}{GH}; \quad \therefore \frac{AB^2}{CD^2}=\frac{EF^2}{GH^2}.$$

But the fig. KAB : the fig. LCD $=$ AB^2 : CD^2; *Theor.* 73.

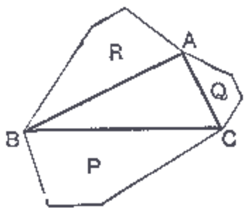
and the fig. MF : the fig. NH $=$ EF^2 : GH^2.

$\therefore$ the fig. KAB : the fig. LCD $=$ the fig. MF : the fig. NH.

Hence if four straight lines are proportional, and a pair of similar rectilineal figures are similarly described on the first and second, and also a pair on the third and fourth, these figures are proportional.

Theorem 74. [Euclid VI. 31]

In a right-angled triangle, any rectilineal figure described on the hypotenuse is equal to the sum of the two similar and similarly described figures on the sides containing the right angle.



Let ABC be a right-angled triangle of which BC is the hypotenuse; and let P, Q, R be similar and similarly described figures on BC, CA, AB respectively.

It is required to prove that

the fig. R + the fig. Q = the fig. P.

Proof. Since AB and BC are corresponding sides of the similar figs. R and P,

$$\therefore \frac{\text{fig. R}}{\text{fig. P}} = \frac{AB^2}{BC^2}. \ldots\ldots\ldots \text{(i)} \qquad \textit{Theor. 73.}$$

In like manner,

$$\frac{\text{fig. Q}}{\text{fig. P}} = \frac{AC^2}{BC^2}. \ldots\ldots\ldots \text{(ii)}$$

Adding the equal ratios on each side in (i) and (ii)

$$\frac{\text{fig. R} + \text{fig. Q}}{\text{fig. P}} = \frac{AB^2 + AC^2}{BC^2}.$$

But $AB^2 + AC^2 = BC^2$; *Theor. 29.*

∴ the fig. R + the fig. Q = the fig. P. Q.E.D.

Corollary. *The area of a circle drawn on the hypotenuse of a right-angled triangle as diameter is equal to the sum of the circles similarly drawn on the other sides.*

Theorem 75. [Euclid III. 35 and 36]

If any two chords of a circle cut one another internally or externally, the rectangle contained by the segments of one is equal to the rectangle contained by the segments of the other.

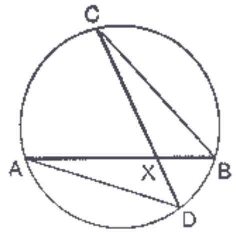


Fig. 1

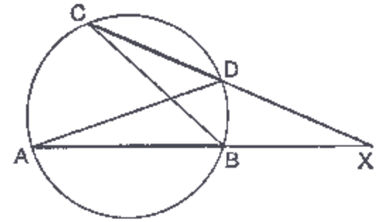


Fig. 2

In the ⊙ABC, let the chords AB, CD cut one another at X, internally in Fig. 1, and externally in Fig. 2.

It is required to prove in both cases that

the rect. XA, XB = *the rect.* XC, XD.

Join AD, BC.

Proof. In the △s AXD, CXB,

the ∠AXD = the ∠CXB, being opp. vert. ∠s in Fig. 1, and the same angle in Fig. 2;

and the ∠A = the ∠C, being ∠s at the Oce, standing on the same arc BD;

∴ the remaining angles are equal; *Theor. 16.*

hence the △s AXD, CXB are equiangular,

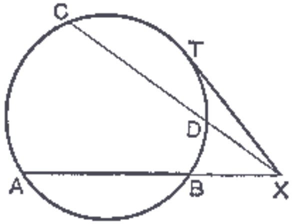
$$\therefore \frac{XA}{XC} = \frac{XD}{XB};$$

$$\therefore XA.XB = XC.XD:$$

that is, the rect. XA, XB = the rect. XC, XD.

Q.E.D.

Corollary. *If from an external point a secant and a tangent are drawn to a circle, the rectangle contained by the whole secant and the part of it outside the circle is equal to the square on the tangent.*



Let XBA be a secant, and XT a tangent drawn to the ⊙ABC from the point X.

It is required to prove that $XA . XB = XT^2$.

Let XDC be a second secant;

then $XA . XB = XC . XD$, *Theor.* 75. Fig. 2.

and this is true for all positions of the line XDC.

Now let XDC turn about X away from the centre, so that the points C and D continually approach one another and ultimately coincide at T;

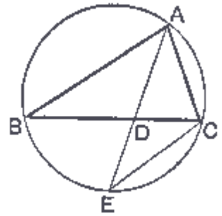
then XDC becomes the tangent XT,

and XC . XD becomes XT . XT, or XT^2,

∴, ultimately, $XA . XB = XT^2$.

* Theorem 76

If the vertical angle of a triangle is bisected by a straight line which cuts the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the segments of the base, together with the square on the straight line which bisects the angle.



Let ABC be a triangle, having the ∠BAC bisected by AD.

It is required to prove that

the rect. AB, AC = *the rect.* BD, DC + *the sq. on* AD.

Suppose a circle circumscribed about the △ABC; and let AD be produced to meet the $\bigcirc^{ce}$ at E.

Join EC.

Proof. Then in the △s BAD, EAC,

because the ∠BAD = the ∠EAC,

and the ∠ABD = the ∠AEC in the same segment;

∴ the remaining ∠BDA = the remaining ∠ECA;

that is, the △s BAD, EAC are equiangular to one another;

$$\therefore \frac{AB}{AE} = \frac{AD}{AC}. \qquad \textit{Theor. } 62.$$

Hence $AB . AC = AE . AD$

$= (AD + DE)AD$

$= AD^2 + AD . DE.$

But $AD . DE = BD . DC$; *Theor.* 75.

∴ the rect. AB, AC = the rect. BD, DC + the sq. on AD.

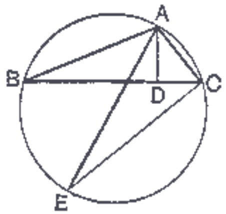
Q.E.D.

EXERCISE

If the vertical angle BAC is bisected *externally* by AD, shew that $AB . AC = BD . DC - AD^2$.

Theorem 77

If from the vertical angle of a triangle a straight line is drawn perpendicular to the base, the rectangle contained by the sides of the triangle is equal to the rectangle contained by the perpendicular and the diameter of the circum-circle.



In the $\triangle$ABC, let AD be the perp. from A to the base BC; and let AE be a diameter of the circum-circle.

It is required to prove that

the rect. AB, AC = *the rect.* AE, AD.

Join EC.

Proof. Then in the $\triangle^s$ BAD, EAC,

the rt. angle BDA = the rt. angle, ECA, in the semicircle ECA,

and the $\angle$ABD = the $\angle$AEC, in the same segment;

$\therefore$ the remaining $\angle$BAD = the remaining $\angle$EAC;

that is, the $\triangle^s$ BAD, EAC are equiangular to one another.

$$\therefore \frac{AB}{AE} = \frac{AD}{AC}; \qquad \textit{Theor. } 62.$$

Hence AB . AC = AE . AD;

or the rect. AB, AC = the rect. AE, AD.

Q.E.D.

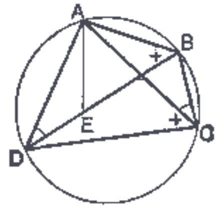
Note. Let a, b, c denote the sides of the $\triangle$ABC, R its circum-radius and p the perp. AD.

Then since

$$AE . AD = AB . AC,$$
$$2R . p = cb;$$
$$\therefore R = \frac{bc}{2p}$$
$$= \frac{abc}{2ap} = \frac{abc}{4\Delta}.$$

Theorem 78. [Ptolemy's Theorem]

The rectangle contained by the diagonals of a quadrilateral inscribed in a circle is equal to the sum of the two rectangles contained by its opposite sides.



Let ABCD be a quadrilateral inscribed in a circle, and let AC, BD be its diagonals.

It is required to prove that

the rect. AC, BD = *the rect.* AB, CD + *the rect.* BC, DA.

Make the $\angle$DAE equal to the $\angle$BAC;

to each add the $\angle$EAC,

then the $\angle$DAC = the $\angle$EAB.

Proof. Then in the $\triangle^s$ EAB, DAC,

the $\angle$EAB = the $\angle$DAC,

and the $\angle$ABE = the $\angle$ACD in the same segment;

$\therefore$ the $\triangle^s$EAB, DAC are equiangular to one another; *Theor.* 16.

$$\therefore \frac{BA}{CA} = \frac{BE}{CD}; \qquad \textit{Theor. } 62.$$

hence AB . CD = AC . BE. ....(i)

Again in the $\triangle^s$ DAE, CAB,

the $\angle$DAE = the $\angle$CAB,

and the $\angle$ADE = the $\angle$ACB, in the same segment;

$\therefore$ the $\triangle^s$ DAE, CAB are equiangular to one another;

$$\therefore \frac{DA}{CA} = \frac{DE}{CB};$$

hence BC . DA = AC . DE ....(ii)

Adding the equal rectangles on each side in (i) and (ii)

$$AB . CD + BC . DA = AC . BE + AC . DE$$
$$= AC (BE + DE)$$
$$= AC . BD.$$

Q.E.D.