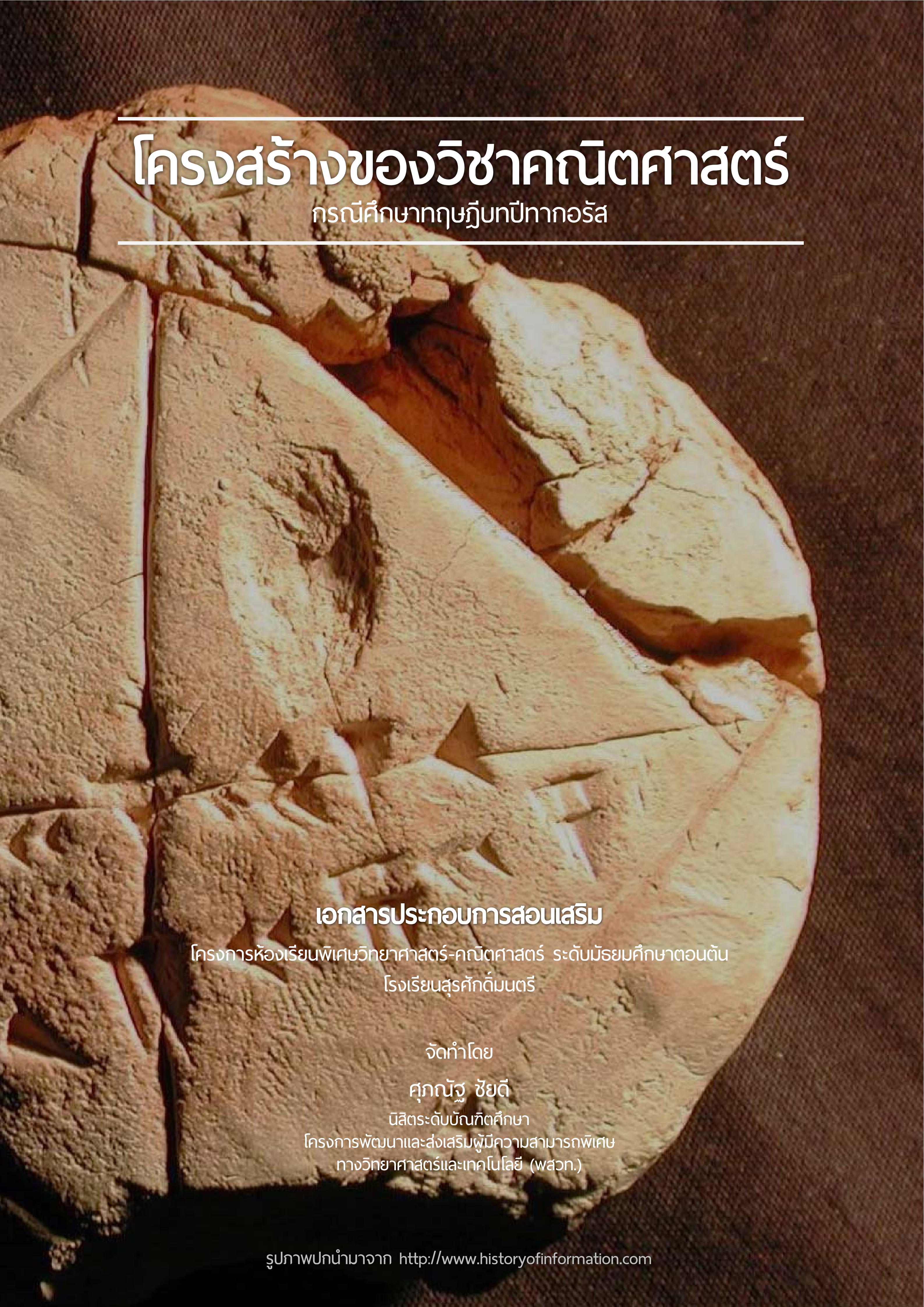


โครงสร้างของวิชาคณิตศาสตร์

กรณีศึกษาทฤษฎีบทปีทากอรัส

เอกสารประกอบการสอนเสริม

โครงการห้องเรียนพิเศษวิทยาศาสตร์-คณิตศาสตร์ ระดับมัธยมศึกษาตอนต้น

โรงเรียนสุรศักดิ์มนตรี

จัดทำโดย

ศุภณัฐ ชัยดี

นิสิตระดับบัณฑิตศึกษา

โครงการพัฒนาและส่งเสริมผู้มีความสามารถพิเศษ

ทางวิทยาศาสตร์และเทคโนโลยี (พสวท.)

รูปภาพปกนำมาจาก http://www.historyofinformation.com

โครงสร้างของวิชาคณิตศาสตร์ : กรณีศึกษาทฤษฎีบทปีทากอรัส

ศุภณัฐ ชัยดี
นิสิตระดับบัณฑิตศึกษา
โครงการพัฒนาและส่งเสริมผู้มีความสามารถพิเศษทางวิทยาศาสตร์และเทคโนโลยี (พสวท.)

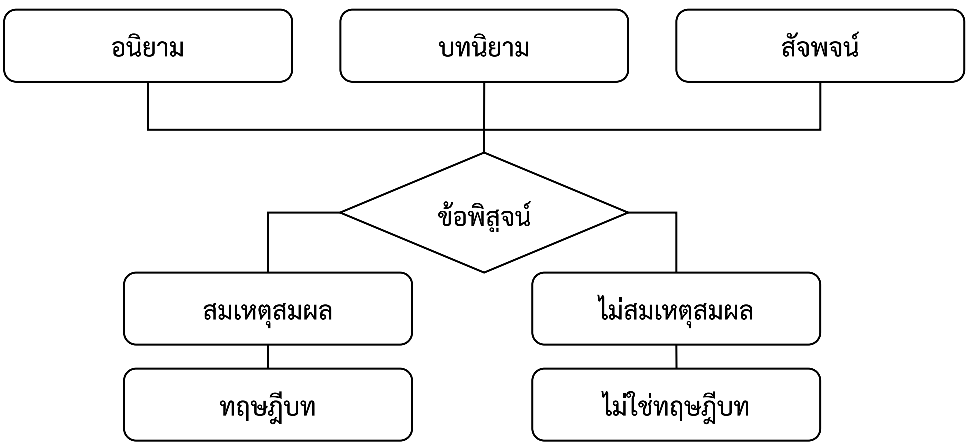
คณิตศาสตร์เป็นวิชาที่ว่าด้วยเรื่องของเหตุ – ผล การได้มาซึ่งทฤษฎีบทหรือข้อความจริงทางคณิตศาสตร์นั้นเริ่มจากสร้างข้อตกลงร่วมกันก่อน จากนั้นจะใช้เหตุผลหรือตรรกะในการสรุปผลให้เป็นไปอย่างสมเหตุสมผล ในบทเรียนนี้เราจะศึกษาโครงสร้างของวิชาคณิตศาสตร์ โดยเริ่มต้นจะให้เห็นภาพของการให้เหตุผลโดยทั่วไปก่อน จากนั้นจะใช้กรณีศึกษาจากทฤษฎีบทที่นักเรียนคุ้นเคยในชั้นเรียนปกติ คือ ทฤษฎีบทปีทากอรัส ซึ่งเราจะศึกษาและวิเคราะห์รูปแบบการพิสูจน์ทฤษฎีบทปีทากอรัสหลายๆ แบบที่แตกต่างกัน เพื่อนำไปสู่ความเข้าใจในการให้เหตุผลทางคณิตศาสตร์ต่อไป

1. โครงสร้างของวิชาคณิตศาสตร์

คณิตศาสตร์ (Mathematics) ประกอบด้วยส่วนประกอบที่สำคัญ 4 ส่วน คือ อนิยาม นิยาม สัจพจน์ และทฤษฎีบท

1. *อนิยาม (Undefined Terms)* หมายถึง คำหรือข้อความที่มีการตกลงกันว่าไม่ต้องให้ความหมาย หรือคำจำกัดความ แต่เข้าใจตรงกันเป็นสากล
2. *บทนิยาม (Defined Terms)* หมายถึง คำหรือข้อความที่มีการให้ความหมายหรือคำจำกัดความไว้อย่างชัดเจน เพื่อทุกคนจะได้มีความเข้าใจที่ตรงกัน
3. *สัจพจน์ (Axiom / postulate)* หมายถึง ข้อความที่ตกลงกันและยอมรับว่าเป็นความจริงโดยไม่ต้องพิสูจน์ และนำไปอ้างเพื่อการพิสูจน์ข้อความอื่นว่าเป็นความจริงได้
4. *ทฤษฎีบท (Theorem)* หมายถึง ข้อความที่ยอมรับว่าเป็นความจริงโดยที่ข้อความเหล่านี้ได้มีการพิสูจน์โดยอาศัยจากทฤษฎีบท นิยาม สัจพจน์ และวิธีการอย่างมีเหตุผล และข้อพิสูจน์นั้นเป็นการอ้างเหตุผลที่สมเหตุสมผล

แผนภาพแสดงความสัมพันธ์ระบบคณิตศาสตร์



ตัวอย่างการพิสูจน์

<u>บทนิยาม</u> ให้ m, n เป็นจำนวนเต็ม เรากล่าวว่า

1. n เป็นจำนวนคู่ ก็ต่อเมื่อ มีจำนวนเต็ม k ที่ทำให้ $n = 2k$
2. n เป็นจำนวนคี่ ก็ต่อเมื่อ มีจำนวนเต็ม k ที่ทำให้ $n = 2k + 1$

งานร่วมปฏิบัติ

1. จงพิสูจน์ว่า สำหรับจำนวนเต็มบวกใด ๆ ถ้า x เป็นจำนวนคู่แล้ว $x + 2$ เป็นจำนวนคู่

2. จงพิสูจน์ว่า สำหรับทุกจำนวนเต็ม n ถ้า ถ้า n เป็นจำนวนคู่แล้ว n^2 เป็นจำนวนคู่

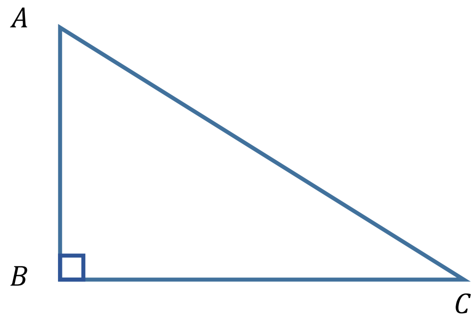
2. ทฤษฎีบทปีทากอรัส

▶ ทบทวนส่วนประกอบของรูปสามเหลี่ยมมุมฉาก

สมมติให้รูปสามเหลี่ยม ABC เป็นสามเหลี่ยมมุมฉาก ที่มีมุม $\hat{B}$ เป็นมุมฉาก

เรียกด้าน $\overline{AB}, \overline{BC}$ ว่า ด้านประกอบมุมฉาก (Cathetus)

เรียกด้าน $\overline{AC}$ ว่า ด้านตรงข้ามมุมฉาก(Hypotenuse)



จะสังเกตว่า ในรูปสามเหลี่ยมมุมฉาก ด้านตรงข้ามมุมฉากจะเป็นด้านที่มีความยาวยาวที่สุด

▶ การพิสูจน์ทฤษฎีบทปีทากอรัส

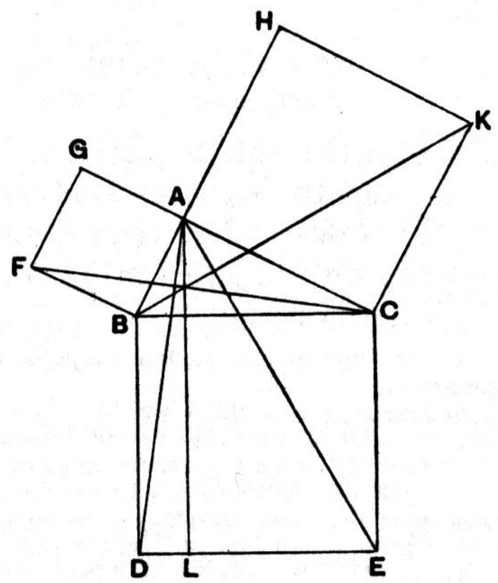
บทตั้ง (Proposition) ต่อไปนี้นำมาจากหนังสือเล่มที่ 1 ในชุด Euclid : The Thirteen Books of The Elements ซึ่งเป็นหนังสือที่วางรากฐานของเรขาคณิตของยูคลิด ดังนี้

Proposition 47

In right-angled triangles the square on the side subtending the right angle is equal to the squares on the sides containing the right angle.

ในรูปสามเหลี่ยมมุมฉากใด ๆ พื้นที่ของรูปสี่เหลี่ยมจัตุรัสบนด้านตรงข้ามมุมฉาก เท่ากับผลบวกของพื้นที่รูปสี่เหลี่ยมจัตุรัสบนด้านประกอบมุมฉาก

ต่อไปนี้จะยกการพิสูจน์ในแบบฉบับดั้งเดิม



Let ABC be a right-angled triangle having the angle BAC right;

I say that the square on BC is equal to the squares on BA, AC.

For let there be described on BC the square $BDEC$, and on BA, AC the squares GB, HC; [I. 46] through A let AL be drawn parallel to either BD or CE, and let AD, FE be joined.

Then, since each of the angles BAC, BAG is right, it follows that with a straight line BA, and at the point A on it, the two straight lines AC, AG not lying on the same side make the adjacent angles equal to two right angles;

therefore CA is in a straight line with AG. [I. 14]

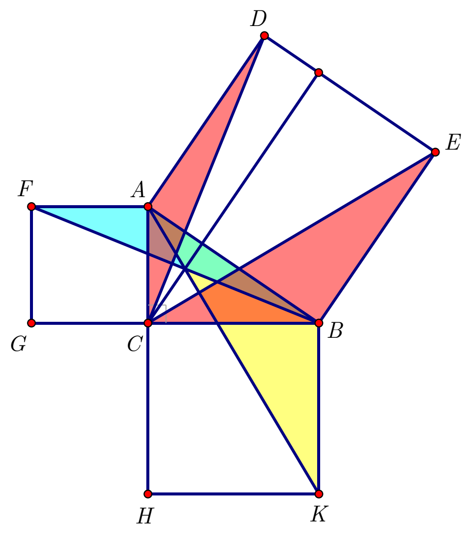
For the same reason BA is also in a straight line with AH.

And, since the angle DBC is equal to the angle FBA: for each is right; let the angle ABC be added to each;

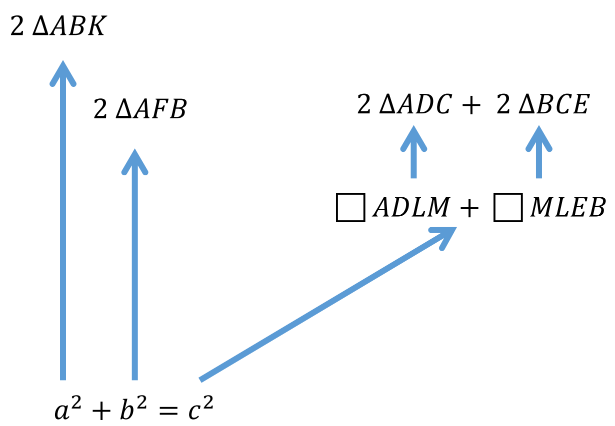
therefore the whole angle DBA is equal to the whole angle FBC. [C. N. 2]

And, since DB is equal to BC, and FB to BA,
the two sides AB, BD are equal to the two sides FB, BC respectively,
and the angle ABD is equal to the angle FBC;
therefore the base AD is equal to the angle FC,
and the triangle ABD is equal to the triangle FBC. [I. 4]
Now the parallelogram BL is double of the triangle ABD, for they have the same base $BD$and are in the same parallels BD, AL.
And the square GB is double of the triangle FBC,
for they again have the same base FB and are in the same parallels FB, GC. [I. 41]
[But the doubles of equals are equal to one another.]
Therefore the parallelogram BL is also equal to the square GB.
Similarly, if AE, BK be joined,
the parallelogram CL can also be proved equal to the square HC;
therefore the whole square $BDEC$ is equal to the two squares GB, HC. [C. N. 2]
And the square $BDEC$ is described on BC, and the squares GB, HC on BA, AC.
Therefore the square on the side BC is equal to the squares on the sides BA, AC.
Therefore etc.

ในหนังสือ “เรขาคณิต” ของโครงการตำราวิทยาศาสตร์และคณิตศาสตร์ มูลนิธิ สอวน ได้ทำการวิเคราะห์การพิสูจน์ และร่างการพิสูจน์ทฤษฎีบทปีทากอรัสฉบับดั้งเดิมไว้ดังนี้



การวิเคราะห์การพิสูจน์



พิสูจน์ พื้นที่ $\square\ CBKH$ หรือ $a^2 = 2\,\Delta ABK$
พื้นที่ $\square\ AFGC$ หรือ $b^2 = 2\,\Delta AFB$

พื้นที่รูปสามเหลี่ยมเท่ากับครึ่งหนึ่งของรูปสี่เหลี่ยม ซึ่งตั้งอยู่บนฐานเดียวกัน ส่วนสูงเท่ากัน

พื้นที่ $\square\ ADLM$ + พื้นที่ $\square\ MLEB = 2\,\Delta ADC + 2\,\Delta BCE$
พื้นที่ $\square\ ABED$ เท่ากับ $c^2 = 2\,\Delta ADC + 2\,\Delta BCE$
แต่ $2\,\Delta ADC = 2\,\Delta AFB$ **
และ $2\,\Delta BCE = 2\,\Delta ABK$
ดังนั้น พื้นที่ $\square\ ABED$ เท่ากับ $c^2 = 2\,\Delta ABK + 2\,\Delta AFB$
$= a^2 + b^2$

** $AD = AB$
$AC = AF$
$DAC = FAB$
ดังนั้น $\Delta ADC \cong \Delta AFB$
และทำนองเดียวกัน
$\Delta BCE \cong \Delta ABK$

งานร่วมปฏิบัติ

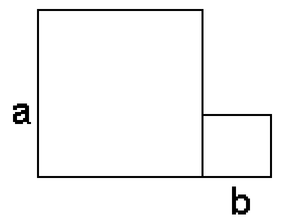
ทฤษฎีบทปีทากอรัส เป็นทฤษฎีบทที่มีผู้พิสูจน์ไว้ในหลายรูปแบบ ในชั้นเรียนนี้ จะให้นักเรียนฝึกวิเคราะห์การพิสูจน์ทฤษฎีบทปีทากอรัสในรูปแบบต่างๆ ที่แตกต่างกัน โดยการพิสูจน์นี้นำมาจาก http://www.cut-the-knot.org/pythagoras/ ซึ่งในเว็บไซต์ดังกล่าวมีการรวบรวมการพิสูจน์ทฤษฎีบทปีทากอรัส 99 วิธี

ให้นักเรียนแบ่งกลุ่ม จากนั้นอ่านและทำความเข้าใจการพิสูจน์ทฤษฎีบทปีทากอรัสในรูปแบบต่างๆ จากนั้นวิเคราะห์การพิสูจน์โดยเริ่มต้นวิเคราะห์ตามขั้นตอนว่า

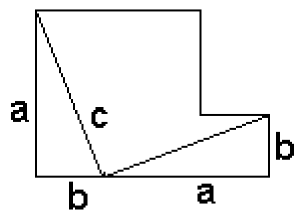
1. จะแสดงว่าอะไร
2. ความรู้ที่จำเป็นต้องใช้มีอะไรบ้าง
3. ลำดับขั้นตอนของการใช้เหตุผล

จากนั้น ให้นักเรียนเขียนการพิสูจน์อย่างเป็นขั้นตอน

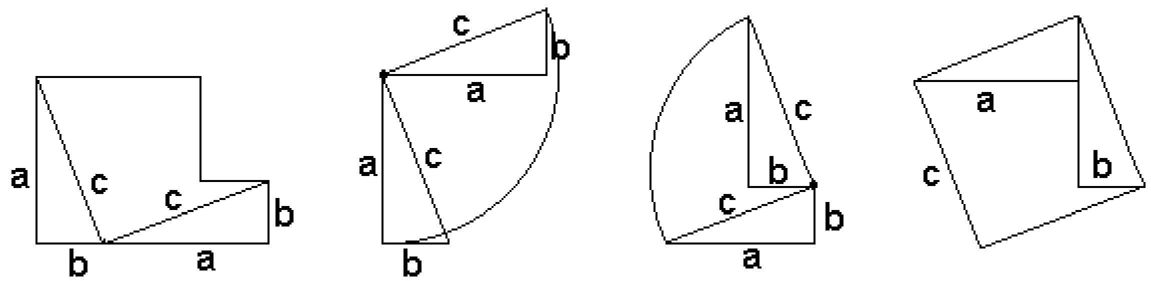
Proof #1



We start with two squares with sides a and b, respectively, placed side by side. The total area of the two squares is a^2+b^2.



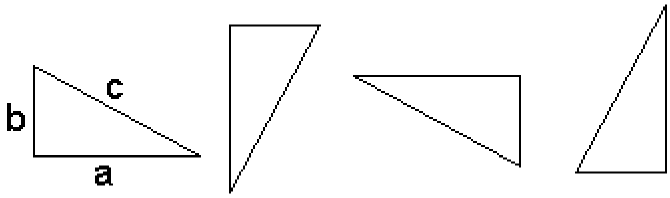
The construction did not start with a triangle but now we draw two of them, both with sides a and band hypotenuse c. Note that the segment common to the two squares has been removed. At this point we therefore have two triangles and a strange looking shape.



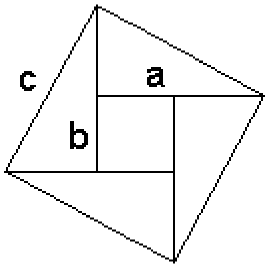
As a last step, we rotate the triangles 90°, each around its top vertex. The right one is rotated clockwise whereas the left triangle is rotated counterclockwise. Obviously the resulting shape is a square with the side c and area c^2. This proof appears in a dynamic incarnation.

(A variant of this proof is found in an extant manuscript by Thâbit ibn Qurra located in the library of Aya Sofya Musium in Turkey, registered under the number 4832. [R. Shloming, Thâbit ibn Qurra and the Pythagorean Theorem, *Mathematics Teacher* 63 (Oct., 1970), 519-528].

Proof #2



Now we start with four copies of the same triangle. Three of these have been rotated 90°, 180°, and 270°, respectively. Each has area ab/2. Let's put them together without additional rotations so that they form a square with side c.

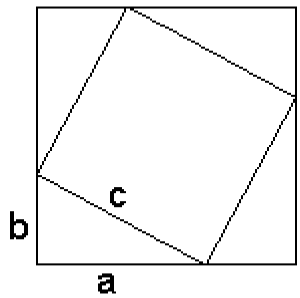


The square has a square hole with the side (a - b). Summing up its area $(a - b)^2$ and 2ab, the area of the four triangles (4·ab/2), we get

$$c^2 = (a - b)^2 + 2ab$$
$$= a^2 - 2ab + b^2 + 2ab$$
$$= a^2 + b^2$$

Proof #3

The fourth approach starts with the same four triangles, except that, this time, they combine to form a square with the side (a + b) and a hole with the side c. We can compute the area of the big square in two ways. Thus



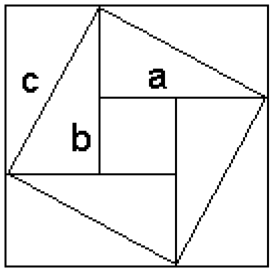
$$(a + b)^2 = 4\cdot ab/2 + c^2$$

simplifying which we get the needed identity.

A proof which combines this with proof #2 is credited to the 12th century Hindu mathematician Bhaskara (Bhaskara II):

Here we add the two identities

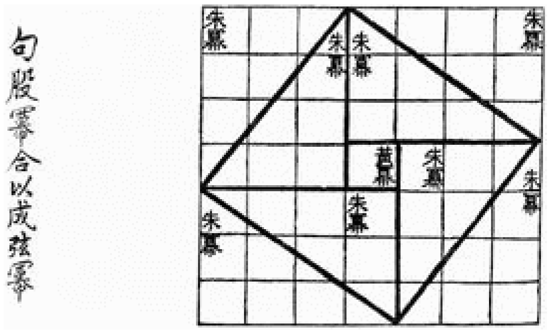
$$c^2 = (a - b)^2 + 4 \cdot ab/2 \text{ and}$$
$$c^2 = (a + b)^2 - 4 \cdot ab/2$$



which gives

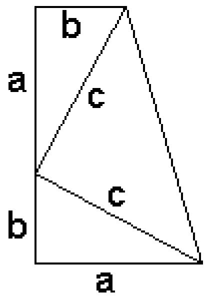
$$2c^2 = 2a^2 + 2b^2.$$

The latter needs only be divided by 2. This is the algebraic proof # 36 in Loomis' collection. Its variant, specifically applied to the 3-4-5 triangle, has featured in the Chinese classic *Chou Pei Suan Ching* dated somewhere between 300 BC and 200 AD and which Loomis refers to as proof 253.



Proof #4

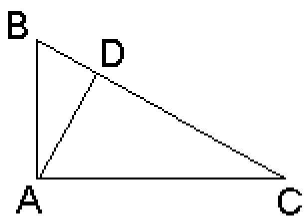
This proof, discovered by President J. A. Garfield in 1876 [Pappas], is a variation on the previous one. But this time we draw no squares at all. The key now is the formula for the area of a trapezoid - *half sum of the bases times the altitude* -$(a + b)/2 \cdot (a + b)$. Looking at the picture another way, this also can be computed as the sum of areas of the three triangles - $ab/2 + ab/2 + c \cdot c/2$. As before, simplifications yield $a^2 + b^2 = c^2$. (There is more to that story.)



Two copies of the same trapezoid can be combined in two ways by attaching them along the slanted side of the trapezoid. One leads to the proof #3.

Proof #5

We start with the original right triangle, now denoted ABC, and need only one additional construct - the altitude AD. The triangles ABC, DBA, and DAC are similar which leads to two ratios:



$$AB/BC = BD/AB \text{ and } AC/BC = DC/AC.$$

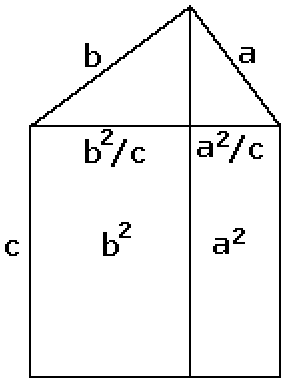
Written another way these become

$$AB \cdot AB = BD \cdot BC \text{ and } AC \cdot AC = DC \cdot BC$$

Summing up we get

AB·AB + AC·AC = BD·BC + DC·BC

= (BD+DC)·BC = BC·BC.



In a little different form, this proof appeared in the *Mathematics Magazine*, 33 (March, 1950), p. 210, in the Mathematical Quickies section, see *Mathematical Quickies*, by C. W. Trigg.
Taking AB = a, AC = b, BC = c and denoting BD = x, we obtain as above

$a^2 = cx$ and $b^2 = c(c - x)$,

which perhaps more transparently leads to the same identity.
In a private correspondence, Dr. France Dacar, Ljubljana, Slovenia, has suggested that the diagram on the right may serve two purposes. First, it gives an additional graphical representation to the present proof #5.

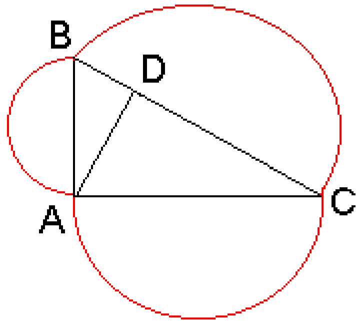
R. M. Mentock has observed that a little trick makes the proof more succinct. In the common notations, c = b cos A + a cos B. But, from the original triangle, it's easy to see that cos A = b/c andcos B = a/c so c = b (b/c) + a (a/c). This variant immediately brings up a question: are we getting in this manner a trigonometric proof? I do not think so, although a trigonometric function (cosine) makes here a prominent appearance. The ratio of two lengths in a figure is a shape property meaning that it remains fixed in passing between similar figures, i.e., figures of the same shape. That a particular ratio used in the proof happened to play a sufficiently important role in trigonometry and, more generally, in mathematics, so as to deserve a special notation of its own, does not cause the proof to depend on that notation. Michael Brozinsky came up with a variant of the proof that I believe could be properly referred to as*lipogrammatic.*

Finally, it must be mentioned that the configuration exploited in this proof is just a specific case of the one from the next proof - Euclid's second and less known proof of the Pythagorean proposition. A separate page is devoted to a proof by the similarity argument.

Proof #6

The next proof is taken verbatim from Euclid VI.31 in translation by Sir Thomas L. Heath. The great G. Polya analyzes it in his *Induction and Analogy in Mathematics* (II.5) which is a recommended reading to students and teachers of Mathematics.

In right-angled triangles the figure on the side subtending the right angle is equal to the similar and similarly described figures on the sides containing the right angle.



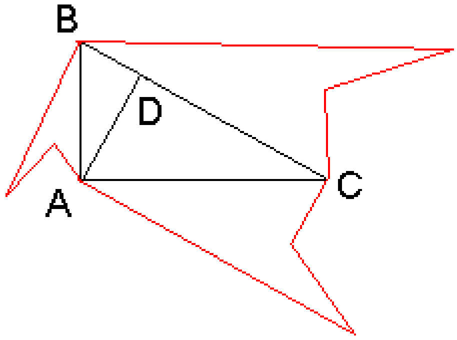
Let ABC be a right-angled triangle having the angle BAC right; I say that the figure on BC is equal to the similar and similarly described figures on BA, AC.

Let AD be drawn perpendicular. Then since, in the right-angled triangle ABC, AD has been drawn from the right angle at A perpendicular to the base BC, the triangles ABD, ADC adjoining the perpendicular are similar both to the whole ABC and to one another [VI.8].

And, since ABC is similar to ABD, therefore, as CB is to BA so is AB to BD [VI.Def.1].

And, since three straight lines are proportional, as the first is to the third, so is the figure on the first to the similar and similarly described figure on the second [VI.19]. Therefore, as CB is to BD, so is the figure on CB to the similar and similarly described figure on BA.

For the same reason also, as BC is to CD, so is the figure on BC to that on CA; so that, in addition, as BC is to BD, DC, so is the figure on BC to the similar and similarly described figures on BA, AC.

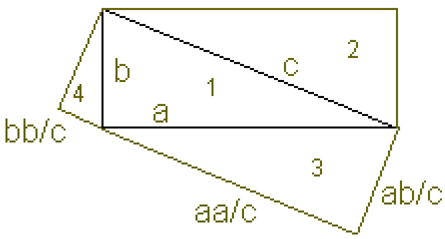


But BC is equal to BD, DC; therefore the figure on BC is also equal to the similar and similarly described figures on BA, AC.

Therefore etc. Q.E.D.

Proof #7

Thus starting with the triangle 1 we add three more in the way suggested in proof #6: similar and similarly described triangles 2, 3, and 4. Deriving a couple of ratios as was done in proof #6 we arrive at the side lengths as depicted on the diagram. Now, it's possible to look at the final shape in two ways:



- as a union of the rectangle (1 + 3 + 4) and the triangle 2, or
- as a union of the rectangle (1 + 2) and two triangles 3 and 4.

Equating the areas leads to

$$ab/c \cdot (a^2 + b^2)/c + ab/2 = ab + (ab/c \cdot a^2/c + ab/c \cdot b^2/c)/2$$

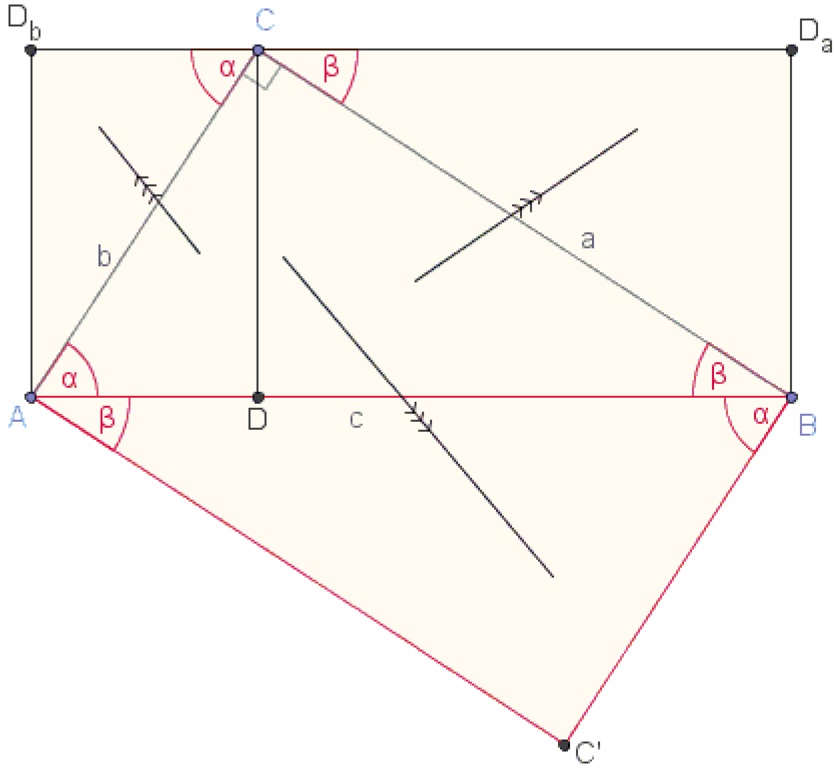
Simplifying we get

$$ab/c \cdot (a^2 + b^2)/c/2 = ab/2, \text{ or } (a^2 + b^2)/c^2 = 1$$

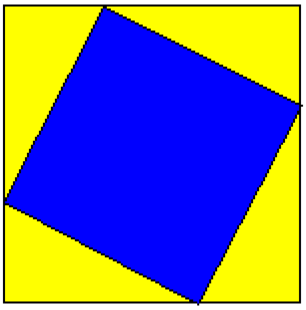
Remark

In hindsight, there is a simpler proof. Look at the rectangle (1 + 3 + 4). Its long side is, on one hand, plain c, while, on the other hand, it's $a^2/c + b^2/c$ and we again have the same identity.

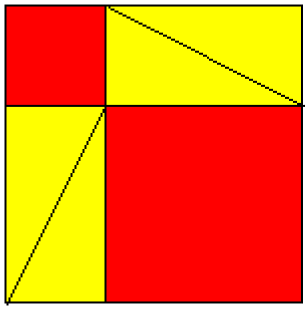
Vladimir Nikolin from Serbia supplied a beautiful illustration:



Proof #8

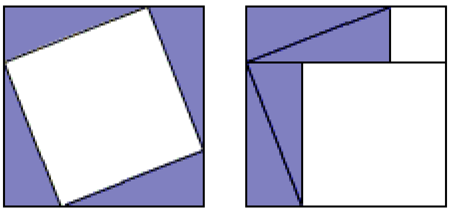


Another proof stems from a rearrangement of rigid pieces, much like proof #1. It makes the algebraic part of proof #3 completely redundant. There is nothing much one can add to the two pictures.



Loomis (pp. 49-50) mentions that the proof "was devised by Maurice Laisnez, a high school boy, in the Junior-Senior High School of South Bend, Ind., and sent to me, May 16, 1939, by his class teacher, Wilson Thornton."

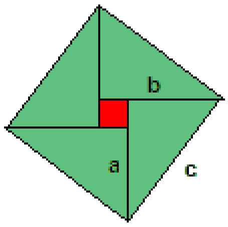
The proof has been published by Rufus Isaac in *Mathematics Magazine*, Vol. 48 (1975), p. 198.



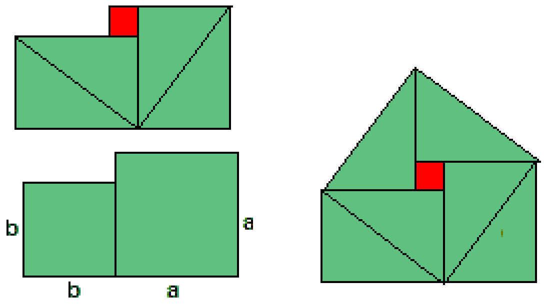
A slightly different rearrangement leads to a hinged dissection illustrated by a Java applet.

R. B. Nelsen reproduces the proof with a remark "based on the one from *Zhou bi suan jing*, a Chinese document dating from approximately 200 BC." Sir Thomas L. Heath mentions it in his commentary (1908) on Euclid I.47 without attribution but with a reference to two other contemporary commentators.

Proof #9

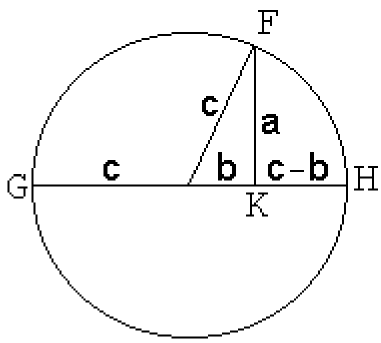


The triangles in Proof #2 may be rearranged in yet another way that makes the Pythagorean identity obvious.



The first two pieces may be combined into one. The result appear in a 1830 book*Sanpo Shinsyo - New Mathematics* - by Chiba Tanehide (1775-1849), [H. Fukagawa, A. Rothman, *Sacred Mathematics: Japanese Temple Geometry*, Princeton University Press, 2008, p. 83].

Proof #10

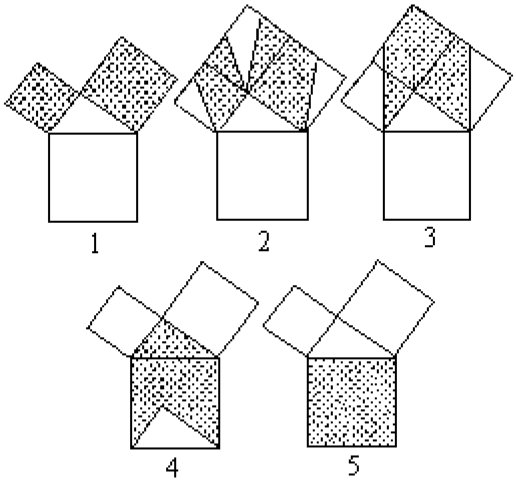


Draw a circle with radius c and a right triangle with sides a and b as shown. In this situation, one may apply any of a few well known facts. For example, in the diagram three points F, G, H located on the circle form another right triangle with the altitude FK of length a. Its hypotenuse GH is split in two pieces: (c + b) and (c - b). So, as in Proof #5, we get $a^2 = (c + b)(c - b) = c^2 - b^2$.

[Loomis, #53] attributes this construction to the great Leibniz, but lengthens the proof about threefold with meandering and misguided derivations.

B. F. Yanney and J. A. Calderhead (*Am Math Monthly*, v.3, n. 12 (1896), 299-300) offer a somewhat different route. Imagine FK is extended to the second intersection F' with the circle. Then, by the Intersecting Chords theorem, FK·KF' = GK·KH, with the same implication.

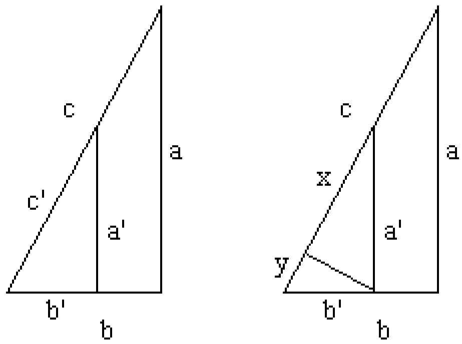
Proof #11



This proof is a variation on the provided example, one of the original Euclid's proofs. In parts 1, 2, and 3, the two small squares are sheared towards each other such that the total shaded area remains unchanged (and equal to a^2+b^2.) In part 3, the length of the vertical portion of the shaded area's border is exactly c because the two leftover triangles are copies of the original one. This means one may slide down the shaded area as in part 4. From here the Pythagorean Theorem follows easily.

(This proof can be found in H. Eves, *In Mathematical Circles*, MAA, 2002, pp. 74-75)

Proof #12



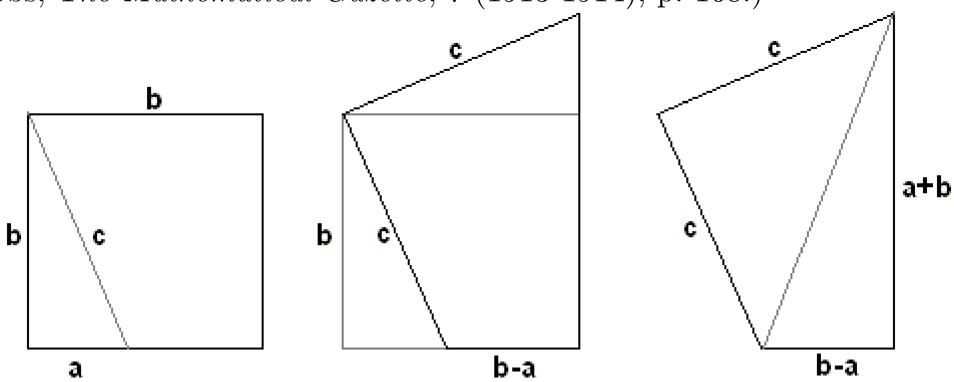
In the diagram there is several similar triangles (abc, a'b'c', a'x, and b'y.) We successively have

$y/b = b'/c$, $x/a = a'/c$, $cy + cx = aa' + bb'$.

And, finally, $cc' = aa' + bb'$. This is very much like Proof #6 but the result is more general.

Proof #13

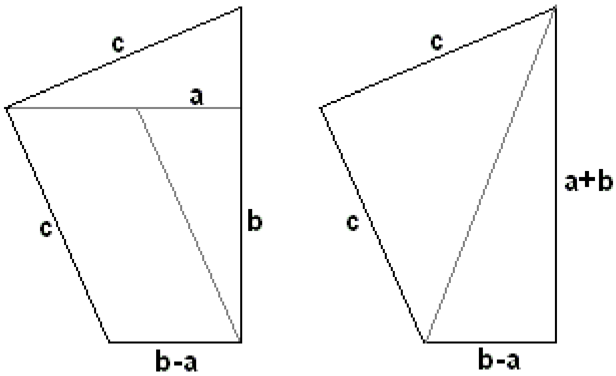
(W. J. Dobbs, *The Mathematical Gazette*, 7 (1913-1914), p. 168.)



This one comes courtesy of Douglas Rogers from his extensive collection. As in Proof #1, the triangle is rotated 90 degrees around one of its corners, such that the angle between the hypotenuses in two positions is right. The resulting shape of area b^2 is then dissected into two right triangles with side lengths (c, c) and (b - a, a + b) and areas $c^2/2$ and $(b - a)(a + b)/2 = (b^2 - a^2)/2$:

$$b^2 = c^2/2 + (b^2 - a^2)/2.$$

J. Elliott adds a wrinkle to the proof by turning around one of the triangles:



Again, the area can be computed in two ways:

$$ab/2 + ab/2 + b(b - a) = c^2/2 + (b - a)(b + a)/2,$$

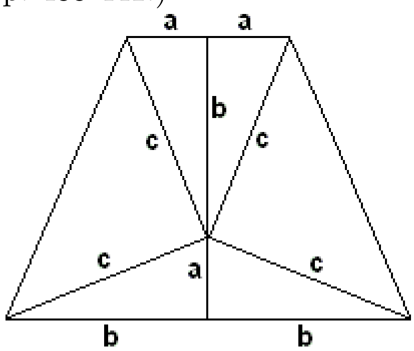
which reduces to

$$b^2 = c^2/2 + (b^2 - a^2)/2,$$

and ultimately to the Pythagorean identity.

Proof #14

This proof, discovered by a high school student, Jamie deLemos (*The Mathematics Teacher*, 88 (1995), p. 79.), has been quoted by Larry Hoehn (*The Mathematics Teacher*, 90 (1997), pp. 438-441.)



On one hand, the area of the trapezoid equals

$$(2a + 2b)/2\cdot(a + b)$$

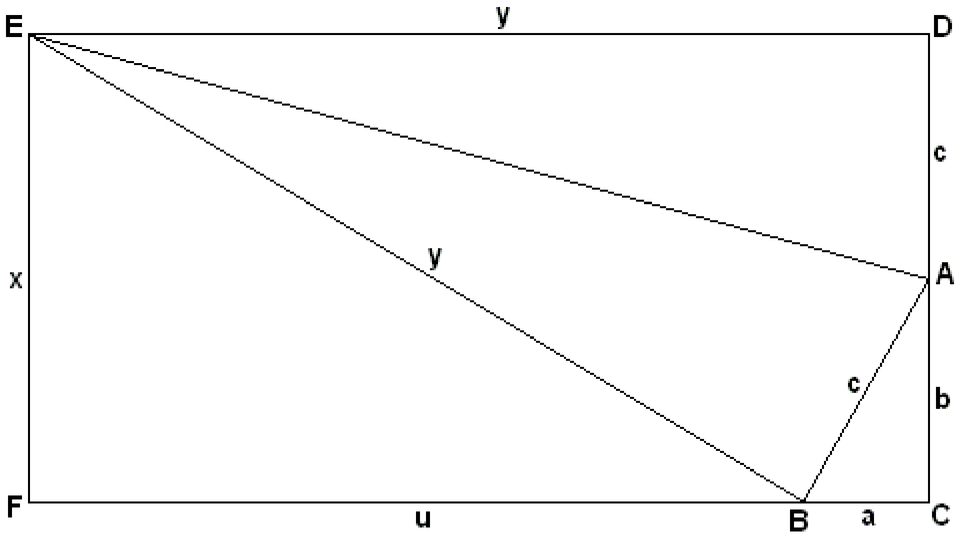
and on the other,

$$2a\cdot b/2 + 2b\cdot a/2 + 2\cdot c^2/2.$$

Equating the two gives $a^2 + b^2 = c^2$.
The proof is closely related to President Garfield's proof.

Proof #15

Larry Hoehn also published the following proof (*The Mathematics Teacher*, 88 (1995), p. 168.):



Extend the leg AC of the right triangle ABC to D so that AD = AB = c, as in the diagram. At D draw a perpendicular to CD. At A draw a bisector of the angle BAD. Let the two lines meet in E. Finally, let EF be perpendicular to CF.

By this construction, triangles ABE and ADE share side AE, have other two sides equal: AD = AB, as well as the angles formed by those sides: ∠BAE = ∠DAE. Therefore, triangles ABE and ADE are congruent by SAS. From here, angle ABE is right.

It then follows that in right triangles ABC and BEF angles ABC and EBF add up to 90°. Thus

$$\angle ABC = \angle BEF \text{ and } \angle BAC = \angle EBF.$$

The two triangles are similar, so that

$$x/a = u/b = y/c.$$

But, EF = CD, or $x = b + c$, which in combination with the above proportion gives

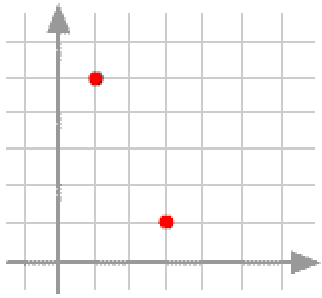
$$u = b(b + c)/a \text{ and } y = c(b + c)/a.$$

On the other hand, $y = u + a$, which leads to

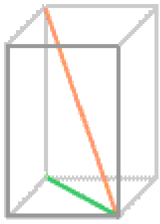
$$c(b + c)/a = b(b + c)/a + a,$$

which is easily simplified to $c^2 = a^2 + b^2$.

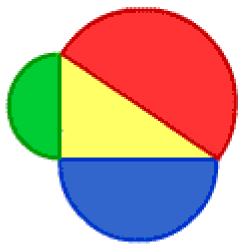
Pythagoras Theorem : Challenging Problems



1. Two points in the x-y-plane have coordinates (1, 5) and (3, 1). Find the distance between them using Pythagoras' theorem. Use the same method to find the distance between the points (347, 999) and (422, 924) *without plotting them.*



2. A rectangular solid has sides of lengths 1, 2 and 3. Use Pythagoras' theorem twice to find the length of its diagonal. How long is the diagonal of a rectangular solid with length l, width w and height h?



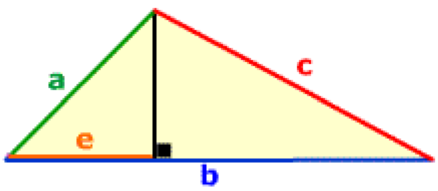
3. Use Pythagoras' theorem to show that, in a right-angled triangle, the area of the semicircle on the hypotenuse is the sum of the areas of the semicircles on the other two sides. Does the same result hold if you replace the semicircles by equilateral triangles?

4. The converse of Pythagoras' theorem is the statement:

"If a triangle has sides of lengths a, b and c and $a^2 + b^2 = c^2$, then the angle opposite the side with length c is a right angle."

In general, the converse of a true statement need not be true. In this case, the converse *is* true, and you can use Pythagoras' theorem to prove so.

In the triangle below, the altitude to the side of length b meets that side at a point a distance e from the vertex. Use Pythagoras' theorem to show that $a^2 + b^2 - 2be = c^2$.



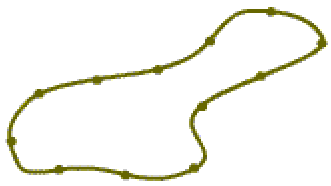
If $a^2 + b^2 = c^2$, what does that tell you about e? What does *that* tell you about the triangle?

In the diagram above, the triangle has an acute angle between the sides of lengths a and b. How would you have to change the argument if that angle were obtuse?

Does a triangle with sides of lengths 13, 5 and 12 have a right angle?

5. A *Pythagorean triple* is a set of three integers which can be lengths of the sides of a right-angled triangle.
   i) Show that {3, 4, 5} is a Pythagorean triple.
   ii) If {a, b, c} is a Pythagorean triple and k is any positive integer, show that {ka, kb, kc} is also a Pythagorean triple.
   iii) If m and n are positive integers with m > n, show that $\{m^2 - n^2, 2mn, m^2 + n^2\}$ is a Pythagorean triple.

6. A landscape gardener has a large loop of cord with 12 equally spaced knots. Explain how she can use that cord and some stakes to lay out the angles for a rectangular flowerbed. [Hint: look at the results of previous problems.]



เอกสารอ้างอิง

1. Thomas L. Heath, **The Thirteen Books of Euclid's Elements (Translated from the text of Heiberg)**, Vol 1, Dover Publication, New York, 1956
2. โครงการตำราวิทยาศาสตร์และคณิตศาสตร์ มูลนิธิ สอวน.**เรขาคณิต** ,กรุงเทพมหานคร :, 2547
3. http://www.cut-the-knot.org/pythagoras/ (สืบค้นวันที่ 14 พฤศจิกายน 2556)