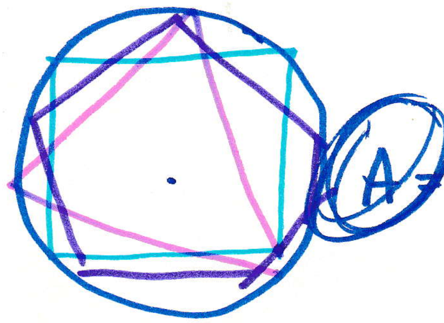
CALCULUS

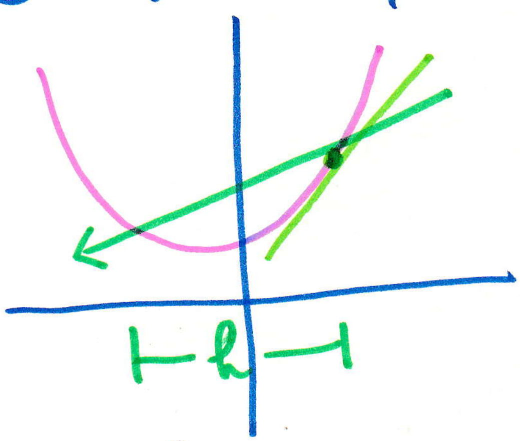


$A = \pi r^2$

ค.ศ~ 17~~ Newton

Leibniz $\Leftrightarrow \frac{dy}{dx}$ / $\int$

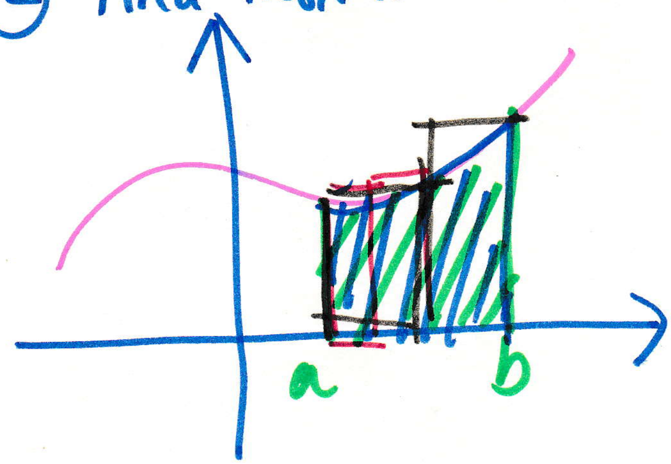
① Tangent line problem

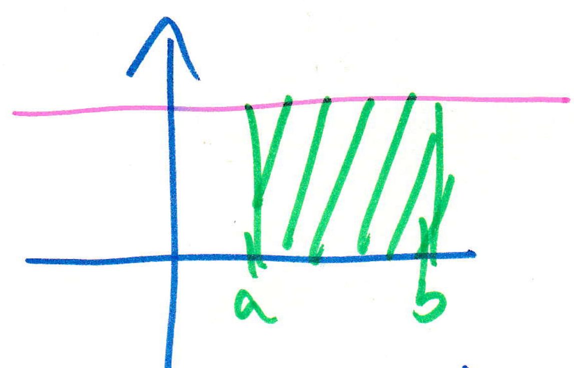


$\lim_{h \to 0}$

differential calculus

② Area Problem.





integral calculus

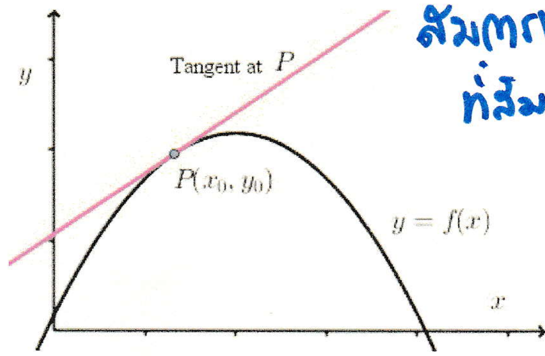
bi bun seki bungakku

微分積分学

Fundamental Theorem of Calculus.

Limits

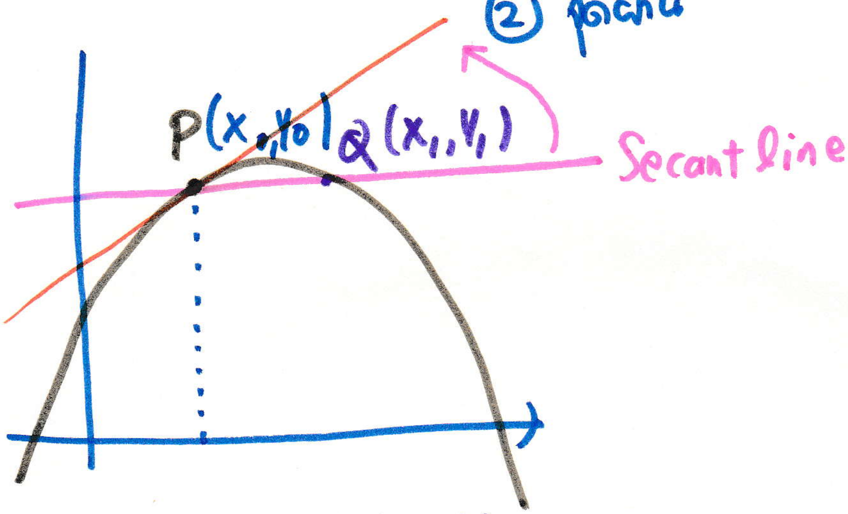
THE TANGENT LINE PROBLEM Given a function f and a point $P(x_0, y_0)$ on the graph of f, find an equation of the line that is tangent to the graph of f at P. (Figure 1.1)



สมการของเส้นตรง
ที่สัมผัสโค้งที่จุด P.

Figure 1.1: A picture of tangent line at point P

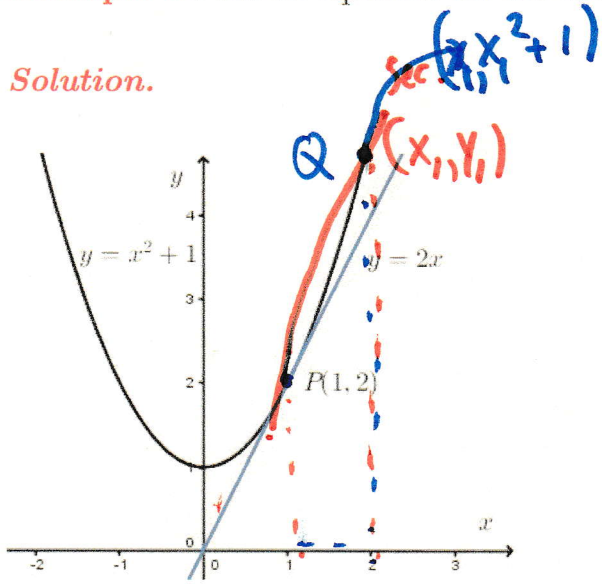
① Slope
② จุดผ่าน



$$m_{sec} = \frac{y_1 - y_0}{x_1 - x_0}$$

Example 1 Find an equation for the tangent line to the parabola $y = x^2 + 1$ at the point $P(1, 2)$.

Solution.



$$m_{sec} = \frac{y_1 - 2}{x_1 - 1} = \frac{(x_1^2+1)-2}{x_1 - 1}$$

พยายามเลื่อน Q ให้ทับกับ P.

$$m_{sec} = \frac{x_1^2 - 1}{(x_1 - 1)}$$

$$m_{sec} = \frac{(x_1 - 1)(x_1 + 1)}{\cancel{(x_1 - 1)}}$$

$$\boxed{m_{sec} = (x_1 + 1)}$$

$$\boxed{\lim_{x_1 \to 1} \frac{x_1^2 - 1}{x_1 - 1}}$$

ย้ายให้ $x = x_1$ ใกล้ๆ $x = 1$

ค่าของ m_{sec} ใกล้ๆ 2

$$\boxed{m_{tan} = 2}$$ ที่ $x_1 = 1$

สมการเส้นตรงที่ผ่าน $P(1,2)$ และมีความชัน $=2$

$$y - 2 = 2(x - 1)$$

$$\boxed{y = 2x}$$

(x_i, y_i), m

$$y - y_i = m(x - x_i)$$

Limits If the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

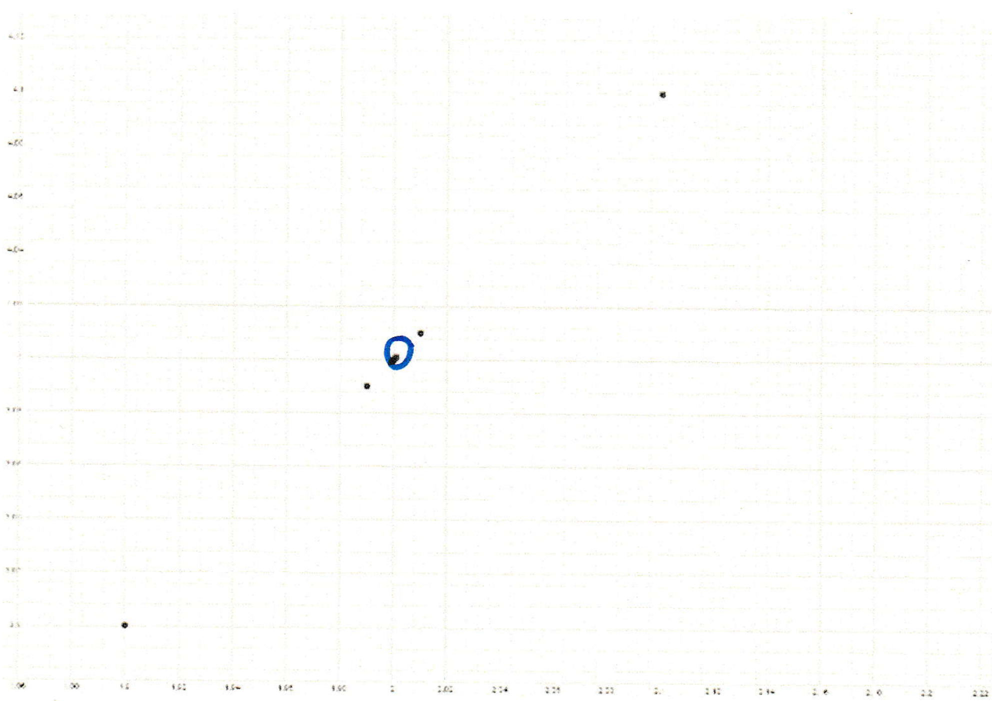
$$\lim_{x \to a} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches a is L", or "$f(x)$ approaches L as x approaches a".

Example 2 Use numerical evidence to make a conjecture about the value of $\displaystyle\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$.

Solution.

	A	B
1	1.9	3.8999999999999998
2	1.99	3.9899999999999979
3	1.999	3.99899999999986
4	1.9999	3.999900000000608
5	1.99999	3.999989999999173
6	1.999999	3.999998999911099
7	1.9999999	3.999999897859482
8		
9	2.0000001	4.00000009769963
10	2.000001	4.0000010000889
11	2.00001	4.00001000000083
12	2.0001	4.000099999999392
13	2.001	4.00100000000014
14	2.01	4.00999999999998
15	2.1	4.1



One-sided Limits If the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but greater than a), then we write

$$\lim_{x \to a^+} f(x) = L$$

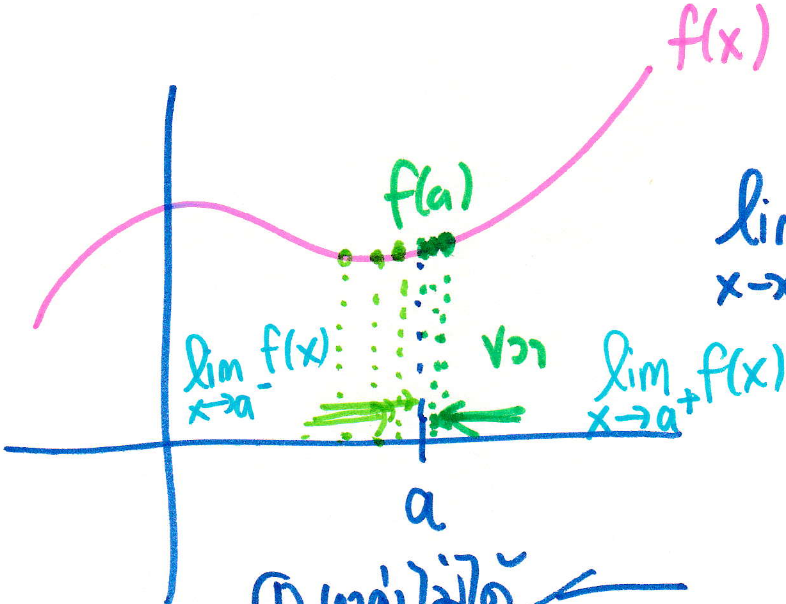
("the limit of $f(x)$ as x approaches a from the right is L" or "$f(x)$ approaches L as x approaches a from the right".)
and if the value of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but less than a), then we write

$$\lim_{x \to a^-} f(x) = L$$

("the limit of $f(x)$ as x approaches a from the left is L" or "$f(x)$ approaches L as x approaches a from the left".)

The Relationship between One-Sided and Two-Sided Limits The two-sided limit of a function $f(x)$ exists at $x = a$ if and only if both of the one-sided limits exist at a and have the same value; that is,

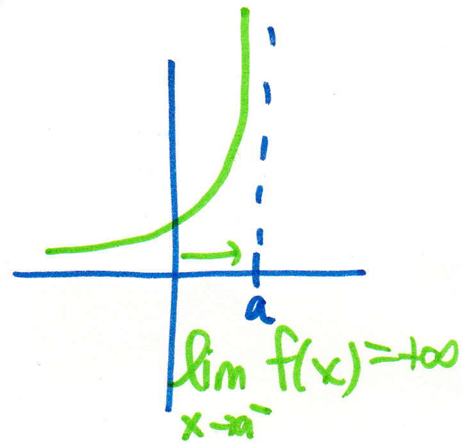
$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$



$\lim_{x \to a} f(x)$

① หาค่าไม่ได้

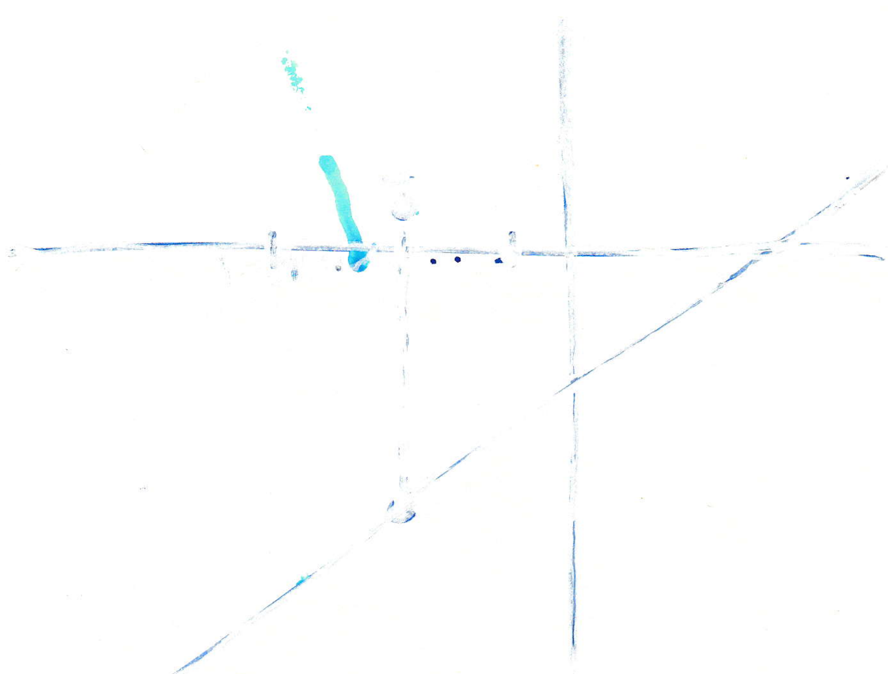
② ไม่มีลิมิต



$$\boxed{\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)}$$

$$\left.\begin{array}{l} \lim_{x \to a^+} f(x) = L_1 \\ \lim_{x \to a^-} f(x) = L_2 \end{array}\right\} L_1 \neq L_2 \Longrightarrow \lim_{x \to a} f(x) \text{ ไม่มีค่า}$$

Limit does not exist.

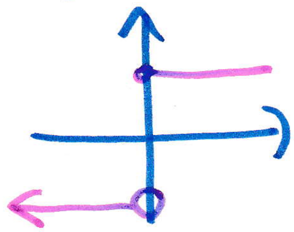


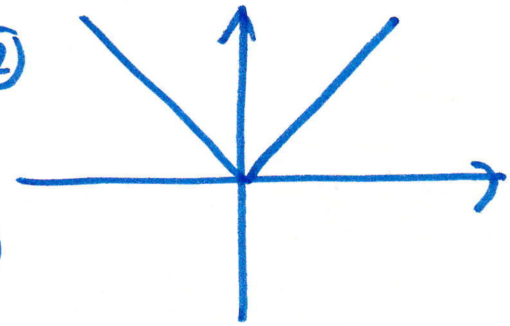
Example 3 Explain why $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Solution.

$$|x| = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$$

$$f(x) = \frac{|x|}{x} = \begin{cases} 1 & ; x \geq 0 \quad (1) \\ -1 & ; x < 0 \quad (2) \end{cases}$$





$$\lim_{x \to 0} \frac{|x|}{x} = \lim_{x \to 0} f(x)$$

$x \geq 0$, $|x| = x$

$\therefore \frac{|x|}{x} = 1$

$x < 0$; $|x| = -x$

$\therefore \frac{|x|}{x} = \frac{-x}{x} = -1$

(1) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$

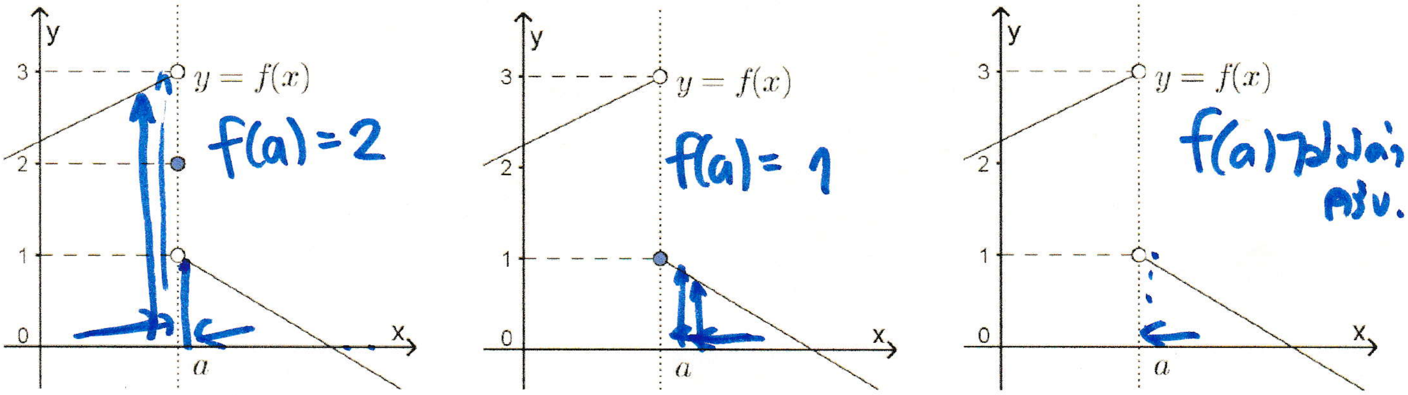
(2) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -1 = -1$

$\because \lim_{x \to 0^+} f(x) \neq \lim_{x \to 0^-} f(x)$

$\therefore \lim_{x \to 0} \frac{|x|}{x}$ ไม่มีค่า

Example 4 For the functions in Figure 1.3, find the one-sided and two-sided limits at $x = a$ if they exist.

Solution.



(a) (b) (c)

(a)
$\lim_{x\to a^+} f(x) = 1$

$\lim_{x\to a^-} f(x) = 3$

$\lim_{x\to a} f(x) =$ ไม่มีค่า ครับ

(b)
$\lim_{x\to a^+} f(x) = 1$

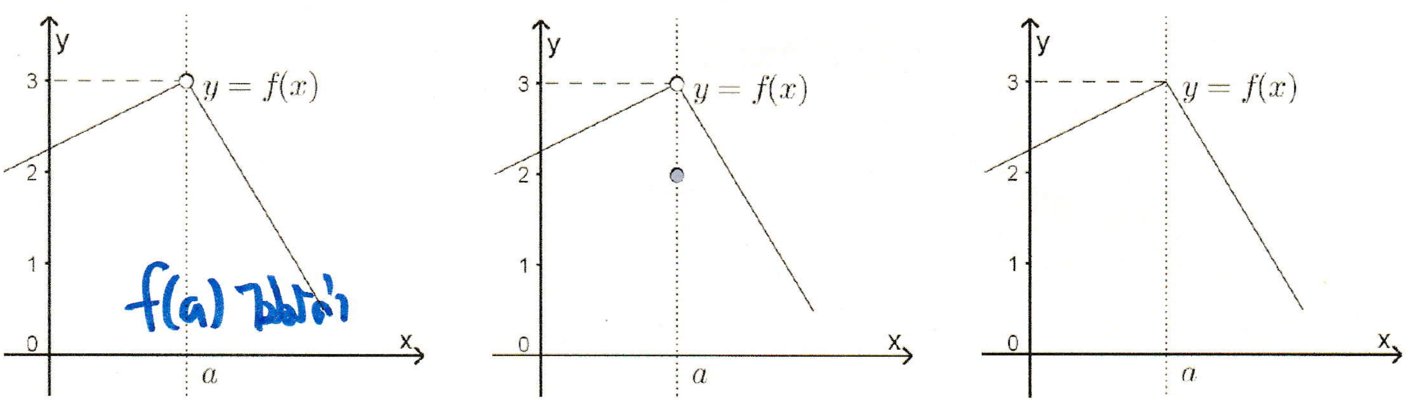
$\lim_{x\to a^-} f(x) = 3$

(c)
$\lim_{x\to a^+} f(x) = 1$

$\lim_{x\to a^-} f(x) = 3$

Example 5 For the functions in Figure 1.4, find the one-sided and two-sided limits at $x = a$ if they exist.

Solution.



(a) (b) (c)

(a)
$\lim_{x\to a^+} f(x) = 3$

$\lim_{x\to a^-} f(x) = 3$

(b)
$\lim_{x\to a^+} f(x) = 3$

$\lim_{x\to a^-} f(x) = 3$

$f(a) = 2$

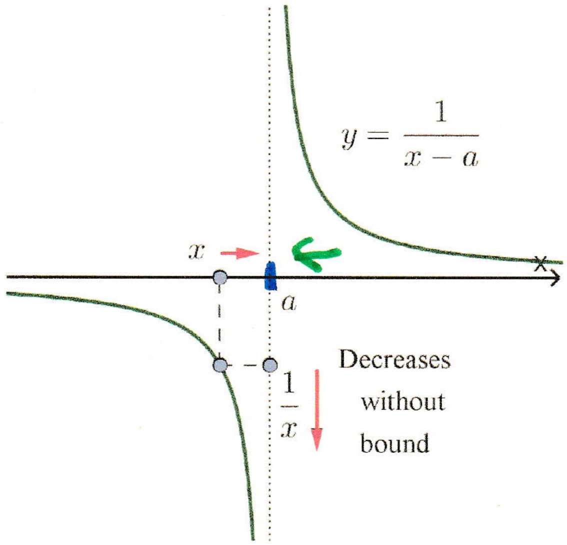
(c)
$\lim_{x\to a^+} f(x) = 3$

$\lim_{x\to a^-} f(x) = 3$

$\lim_{x\to a} f(x) = 3$

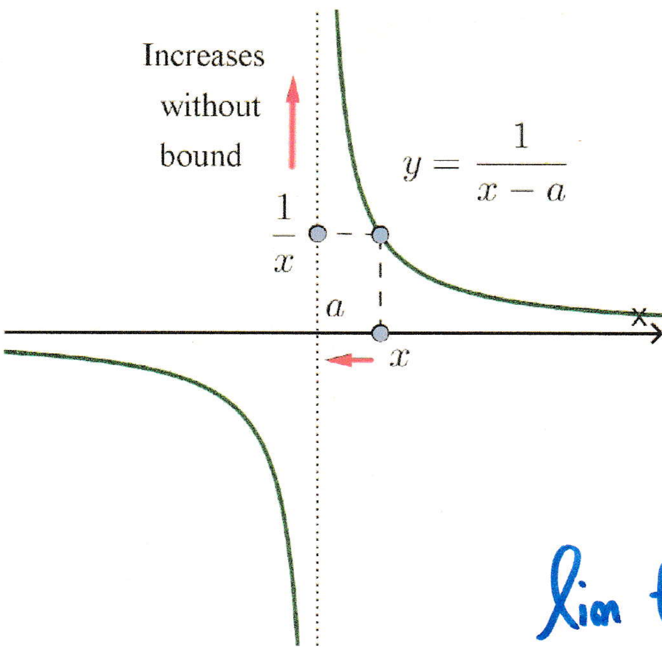
Infinite Limits

x	$a-1$	$a-0.1$	$a-0.01$	$a-0.001$	$a-0.0001$	$\cdots$	a
$\frac{1}{x-a}$	-1	-10	-100	-1000	-10,000	$\cdots$	



$\lim_{x \to a^-} f(x) = -\infty$

x	a	$\cdots$	$a+0.0001$	$a+0.001$	$a+0.01$	$a+0.1$	$a+1$
$\frac{1}{x-a}$		$\cdots$	10,000	1000	100	10	1



$\lim_{x \to a^+} f(x) = +\infty$

$\lim_{x \to a} f(x) =$ ไม่มีค่า

INFINITE LIMITS The expressions

$$\lim_{x \to a^-} f(x) = +\infty \text{ and } \lim_{x \to a^+} f(x) = +\infty$$

denote that $f(x)$ increases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = +\infty.$$

Similarly, the expressions

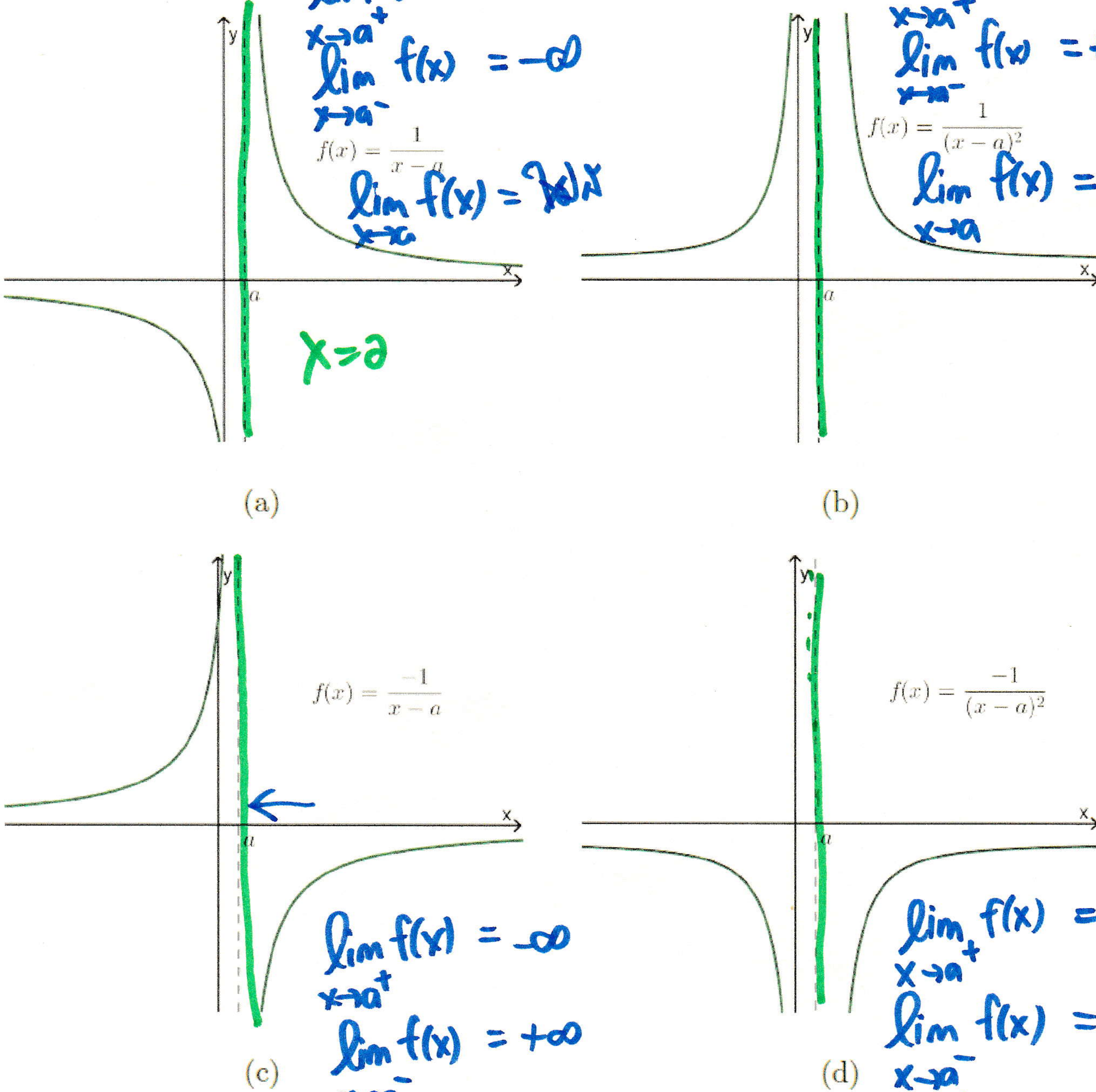
$$\lim_{x \to a^-} f(x) = -\infty \text{ and } \lim_{x \to a^+} f(x) = -\infty$$

denote that $f(x)$ decreases without bound as x approaches a from the left and from the right, respectively. If both are true, then we write

$$\lim_{x \to a} f(x) = -\infty.$$

Example 6 For the functions in Figure 1.7, describe the limits at $x = a$ in appropriate limit notation.

Solution.



(a)
$\lim_{x\to a^+} f(x) = +\infty$
$\lim_{x\to a^-} f(x) = -\infty$
$\lim_{x\to a} f(x) =$ ไม่มี

(b)
$\lim_{x\to a^+} f(x) = +\infty$
$\lim_{x\to a^-} f(x) = +\infty$
$\lim_{x\to a} f(x) = +\infty$

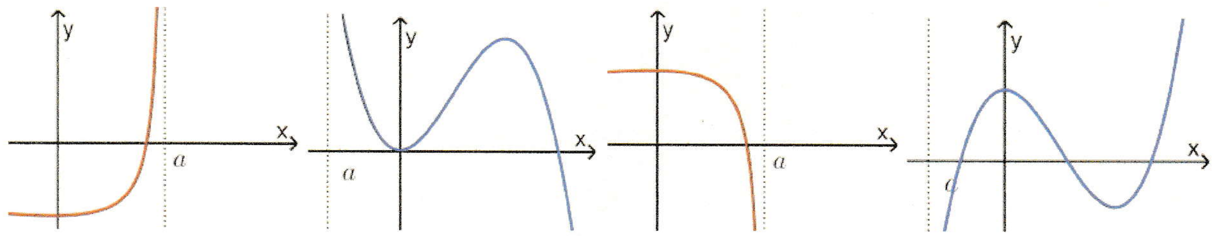
(c)
$\lim_{x\to a^+} f(x) = -\infty$
$\lim_{x\to a^-} f(x) = +\infty$
$\lim_{x\to a} f(x) =$ ไม่มี

(d)
$\lim_{x\to a^+} f(x) = -\infty$
$\lim_{x\to a^-} f(x) = -\infty$
$\lim_{x\to a} f(x) = -\infty$

1.1.3 Vertical Asymptotes เส้นกำกับแนวขึ้น

If the graph of $f(x)$ either rises or falls without bound, squeezing closer and closer to the vertical line $x = a$ as x approaches a from the side indicated in the limit, we call the line $x = a$ *vertical asymptote* of the curve $y = f(x)$.

Figure 1.8 illustrates geometrically what happen when any of the following situations occur:



$\lim_{x\to a^-} f(x) = +\infty$ $\lim_{x\to a^+} f(x) = +\infty$ $\lim_{x\to a^-} f(x) = -\infty$ $\lim_{x\to a^+} f(x) = -\infty$

Figure 1.8: Examples of vertical asymptotes

Sampling Pitfalls

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right) = ?$$

x	$\frac{\pi}{x}$	$f(x) = \sin\left(\frac{\pi}{x}\right)$
$x = \pm 1$	$\pm\pi$	$\sin(\pm\pi) = 0$
$x = \pm 0.1$	$\pm 10\pi$	$\sin(\pm 10\pi) = 0$
$x = \pm 0.01$	$\pm 100\pi$	$\sin(\pm 100\pi) = 0$
$x = \pm 0.001$	$\pm 1000\pi$	$\sin(\pm 1000\pi) = 0$
$x = \pm 0.0001$	$\pm 10{,}000\pi$	$\sin(\pm 10{,}000\pi) = 0$
$\vdots$	$\vdots$	$\vdots$