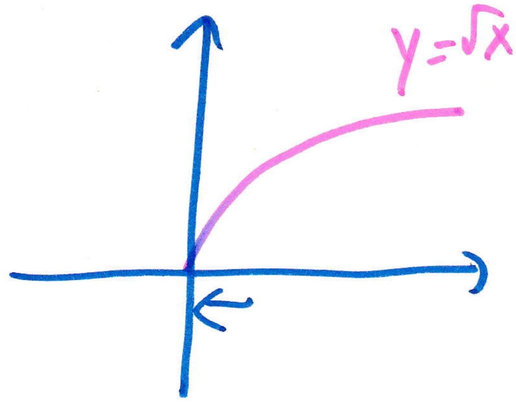
$$\boxed{\lim_{x\to a} f(x) = L \Leftrightarrow \lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)}$$

① Domain ของฟังก์ชัน

$$\boxed{\lim_{x\to 0} \sqrt{x} = 0}$$

$= \cancel{\text{ไม่มีค่า}}$



$f(x) = \sqrt{x} \quad D_f = [0, \infty)$

→ ถ้า D เป็นโดเมนของฟังก์ชันที่มีจุด a เป็นจุดปลายช่วง

ถ้า a เป็นจุดปลายช่วงทางซ้ายแล้ว จะถือว่า

$\lim_{x\to a} f(x)$ คือ $\lim_{x\to a^+} f(x)$

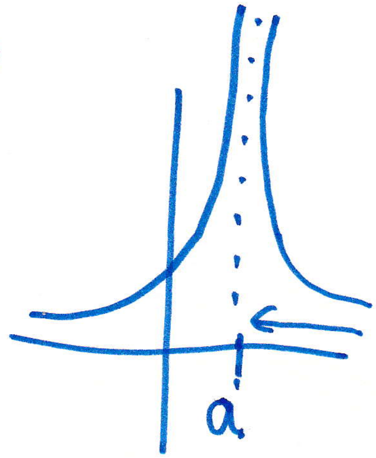
ถ้า a เป็นจุดปลายช่วงทางขวาแล้ว จะถือว่า

$\lim_{x\to a} f(x)$ คือ $\lim_{x\to a^-} f(x)$

(*) CHECK <u>DOMAIN</u>.

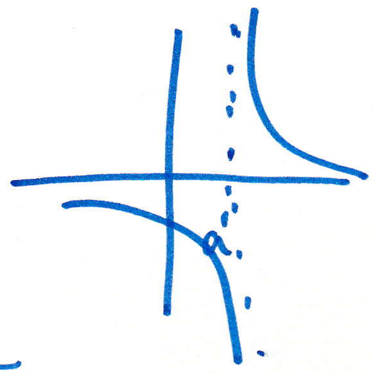
ลิมิตมีค่า $\Rightarrow$ จำนวนจริง L สักตัว

$$\lim_{x \to a} f(x) = +\infty$$

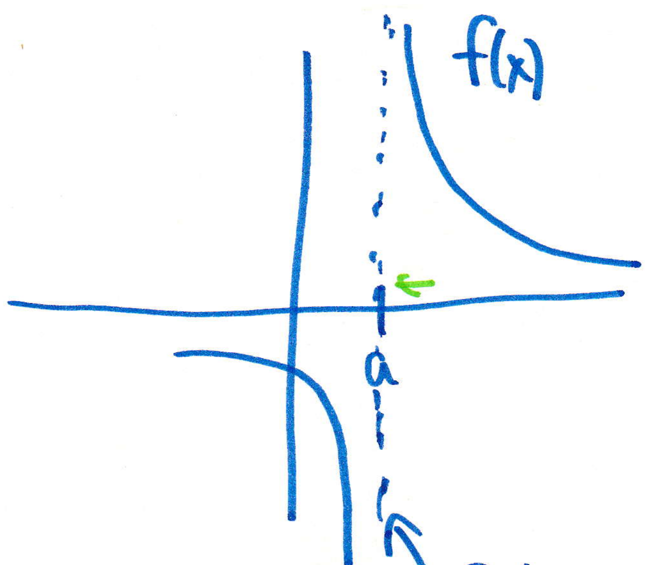


$$\lim_{x \to a^-} f(x) = -\infty$$

$$\lim_{x \to a^+} f(x) = +\infty$$



$\lim_{x \to a} f(x)$ ไม่มีค่า



$$\lim_{x \to a^+} f(x) = +\infty$$

$$\lim_{x \to a^-} f(x) = -\infty$$

เส้นกำกับแนวดิ่ง $\boxed{x = a}$

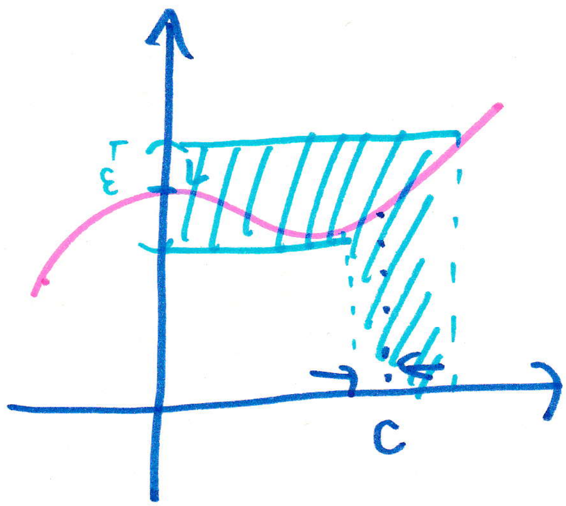
Formal Definition of limit

Let f be a real-valued function defined on $D \subseteq \mathbb{R}$. Let a be a limit point of D and let L be a real number

$\lim_{x \to a} f(x) = L$ **if and only if** $\boxed{\delta - \varepsilon}$

for every $\epsilon > 0$, there is $\delta > 0$ such that for all $x \in D$,

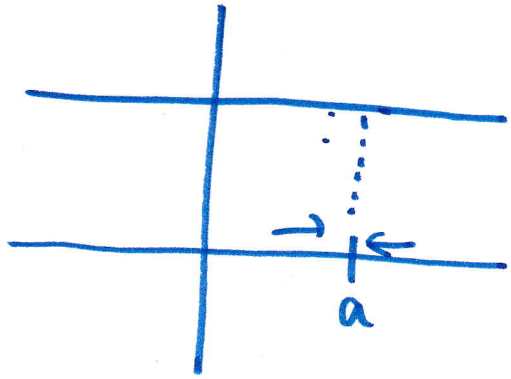
if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$.



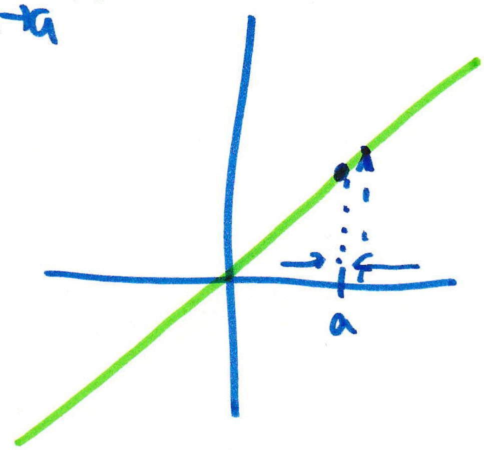
Theorem (The uniqueness of limit)

The limit of a function is unique if it exists.

$\lim_{x \to a} k$ $f(x) = k$



$\lim_{x \to a} x$ $f(x) = x$



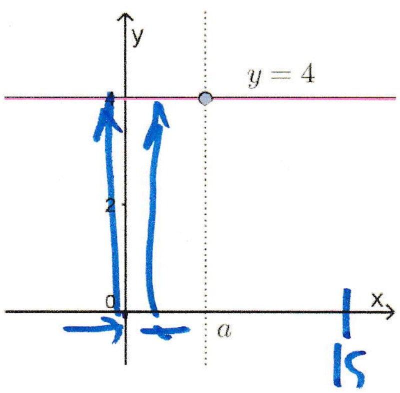
1.2 Computing Limits

1.2.1 Some Basic Limits

THEOREM 1.1 Let a and k be real numbers.

(a) $\lim_{x\to a} k = k$ (b) $\lim_{x\to a} x = a$ (c) $\lim_{x\to 0^-} \frac{1}{x} = -\infty$ (d) $\lim_{x\to 0^+} \frac{1}{x} = +\infty$

Example 7 If $f(x) = k$ is a constant function, then the values of $f(x)$ remain fixed at k as x varies, which explains why $\lim_{x\to a} k = k$ for all value of a. For example,



$\lim_{x\to -5} 4 =$ 4

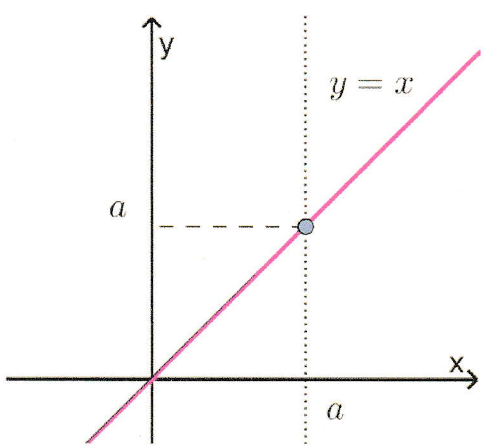
$\lim_{x\to 0} 4 =$ 4

$\lim_{x\to 15} 4 =$ 4

Figure 1.9: Graph of $f(x) = 4$

Example 8 If $f(x) = x$, then the values of $f(x)$ always equals to x varies, which explains why $\lim_{x\to a} x = a$ for all value of a. For example,

Solution.



$\lim_{x\to 0} x =$ 0

$\lim_{x\to 3} x =$ 3

$\lim_{x\to -100} x =$ -100

$\lim_{x\to \pi} x = \pi$

Figure 1.10: Graph of $f(x) = x$

THEOREM 1.2 Let a be a real number, and suppose that

$$\lim_{x\to a} f(x) = L_1 \qquad \text{and} \qquad \lim_{x\to a} g(x) = L_2.$$

That is, the limits exist and have values L_1 and L_2, respectively. Then:

(a) $\lim_{x\to a}[f(x)+g(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) = L_1 + L_2$

(b) $\lim_{x\to a}[f(x)-g(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) = L_1 - L_2$

(c) $\lim_{x\to a}[f(x)g(x)] = \left(\lim_{x\to a} f(x)\right)\left(\lim_{x\to a} g(x)\right) = L_1 L_2$

(d) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} = \dfrac{L_1}{L_2}$, provided $L_2 \neq 0$

(e) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)} = \sqrt[n]{L_1}$, provided $L_1 > 0$ if n is even.

Moreover, these statements are also true for the one-sided limits as $x \to a^-$ or as $x \to a^+$.

THEOREM 1.3 For any polynomial

$$p(x) = c_0 + c_1 x + \cdots + c_n x^n$$

and any real number a,

$$\lim_{x\to a} p(x) = c_0 + c_1 a + \cdots + c_n a^n = p(a).$$

1.2.2 Limits of Polynomials and Rational Functions as $x \to a$

Example 9 Find $\lim_{x\to 2}(x^3 - 3x + 5)$.

Solution.

$$\lim_{x\to 2}(x^3-3x+5) = \lim_{x\to 2} x^3 - \lim_{x\to 2}(3x) + \lim_{x\to 2} 5$$

$$= \lim_{x\to 2} x^3 - 3\lim_{x\to 2} x + \lim_{x\to 2} 5$$

$$= 8 - 3(2) + 5$$

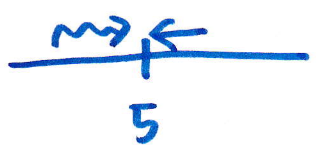
$$= 8 - 6 + 5$$

$$= 7 \quad \#$$

Example 10 Find

(a) $\lim_{x\to 5^+} \dfrac{3-x}{(x-5)(x+1)}$ (b) $\lim_{x\to 5^-} \dfrac{3-x}{(x-5)(x+1)}$ (c) $\lim_{x\to 5} \dfrac{3-x}{(x-5)(x+1)}$.

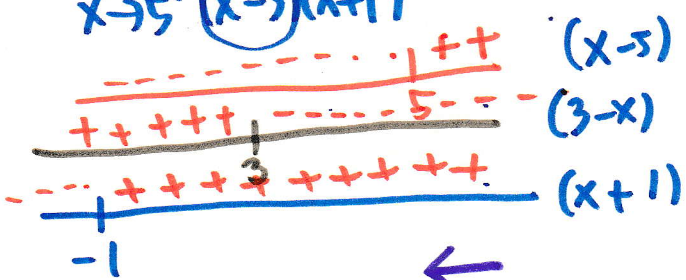
Solution. $+\infty$



$$\lim_{x\to 5^-} \frac{3-x}{(x-5)(x+1)} = +\infty$$

$$\lim_{x\to 5^+} \frac{3-x}{(x-5)(x+1)} = -\infty$$

$$\therefore \lim_{x\to 5} \frac{3-x}{(x-5)(x+1)} \text{ ไม่มีค่า}$$



Limits of Rational Functions $\dfrac{f(x)}{g(x)}$ as $x \to a$ when $\lim\limits_{x\to a} f(x) = 0 = \lim\limits_{x\to a} g(x)$

Example 11 Find

(a) $\displaystyle\lim_{x\to 5} \frac{x^2 - 10x + 25}{x - 5}$ (b) $\displaystyle\lim_{x\to 2} \frac{3x - 6}{x^2 - x - 2}$ (c) $\displaystyle\lim_{x\to 3} \frac{x^2 - 5x + 6}{x^2 - 6x + 9}$.

$\boxed{\dfrac{0}{0}} \leftarrow$ indeterminate form

Solution.

(a) $\displaystyle\lim_{x\to 5} \frac{x^2-10x+25}{x-5} = \lim_{x\to 5} \frac{\cancel{(x-5)}(x-5)}{\cancel{(x-5)}} = \lim_{x\to 5} x-5 = 0$

(b) $\displaystyle\lim_{x\to 2} \frac{3x-6}{x^2-x-2} = \lim_{x\to 2} \frac{3\cancel{(x-2)}}{\cancel{(x-2)}(x+1)} = \lim_{x\to 2} \frac{3}{x+1} = \frac{3}{3} = 1$

(c) $\displaystyle\lim_{x\to 3} \frac{x^2-5x+6}{x^2-6x+9} = \lim_{x\to 3} \frac{\cancel{(x-3)}(x-2)}{\cancel{(x-3)}(x-3)} = \lim_{x\to 3} \frac{x-2}{x-3}$

ซ้าย $\displaystyle\lim_{x\to 3^-} \frac{x-2}{x-3} = -\infty$

ขวา $\displaystyle\lim_{x\to 3^+} \frac{x-2}{x-3} = +\infty$

$\therefore$ ไม่มีลิมิต

Theorem 1.4 Let

$$f(x) = \frac{p(x)}{q(x)}$$

be a rational function, and let a be any real number.

(a) If $q(a) \neq 0$, then $\lim\limits_{x\to a} f(x) = f(a)$.

(b) If $q(a) = 0$ but $p(a) \neq 0$, then $\lim\limits_{x\to a} f(x)$ does not exist.

Limits Involving Radicals

Conjugate : สังยุค

Example 12 Find $\lim\limits_{x\to 4} \dfrac{x-4}{\sqrt{x}-2}$.

Solution.

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} \times \frac{\sqrt{x}+2}{\sqrt{x}+2} = \lim_{x\to 4} \frac{(x-4)(\sqrt{x}+2)}{(\sqrt{x})^2-2^2}$$

$$= \lim_{x\to 4} \frac{\cancel{(x-4)}(\sqrt{x}+2)}{\cancel{x-4}}$$

$$= \lim_{x\to 4} \sqrt{x}+2$$

$$= \sqrt{4}+2$$

$$= 4$$

$$\lim_{x\to 4} \frac{x-4}{\sqrt{x}-2} = \lim_{x\to 4} \frac{(\sqrt{x})^2-(2)^2}{\sqrt{x}-2}$$

$$= \lim_{x\to 4} \frac{\cancel{(\sqrt{x}-2)}(\sqrt{x}+2)}{\cancel{\sqrt{x}-2}}$$

$$= \lim_{x\to 4} \sqrt{x}+2 = 4 \quad \#$$

Limits of Piecewise-Defined Functions

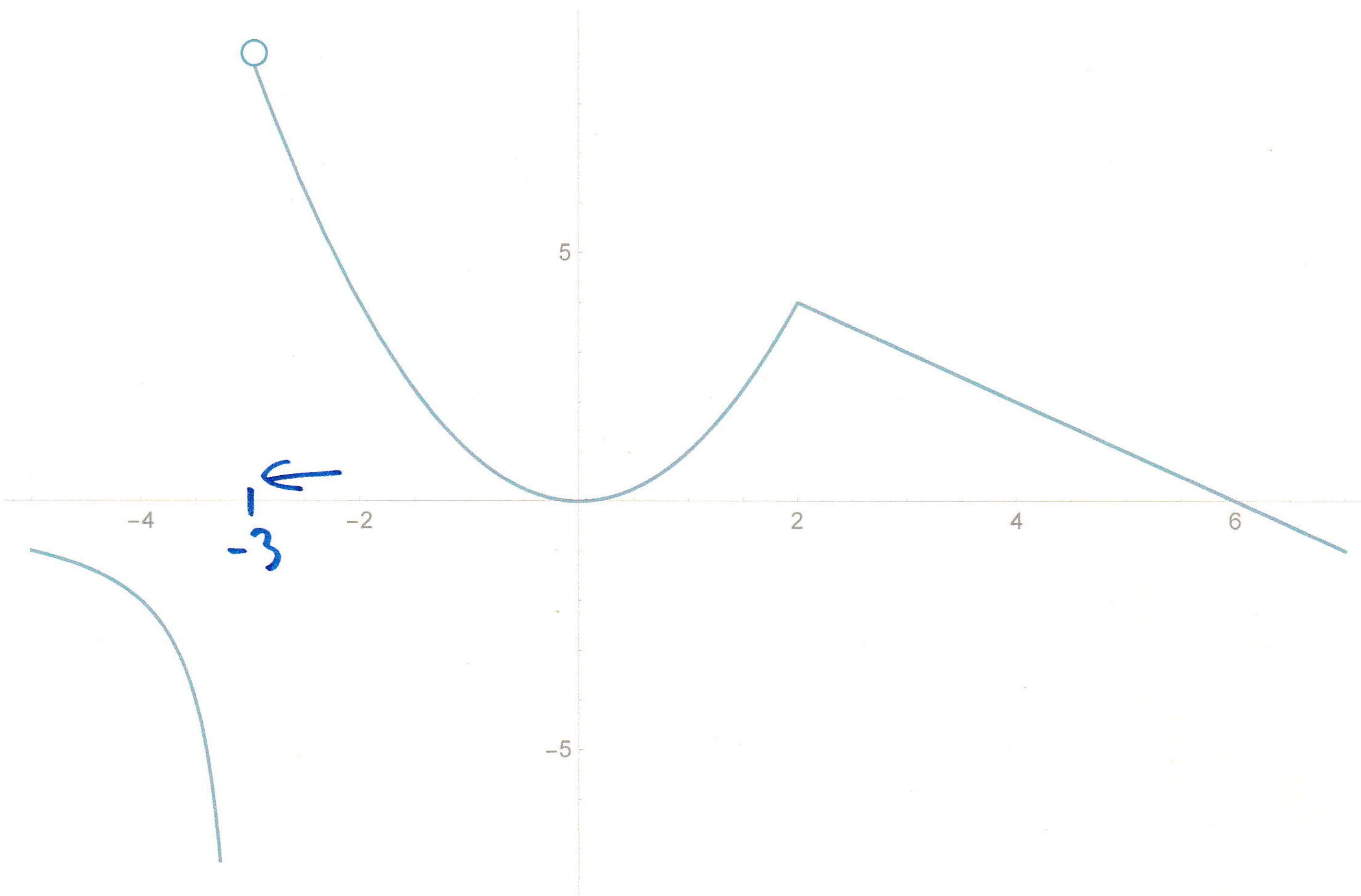
Example 13 Let

$$f(x) = \begin{cases} \dfrac{2}{x+3}, & x < -3 \\ x^2, & -3 < x \leq 2 \\ 6 - x, & x > 2 \end{cases}.$$

Find

(a) $\lim\limits_{x \to -3} f(x)$ (b) $\lim\limits_{x \to 0} f(x)$ (c) $\lim\limits_{x \to 2} f(x)$.

Solution.



(a) $\lim_{x\to -3} f(x) \longrightarrow \lim_{x\to -3^-} f(x) = \lim_{x\to -3^-} \frac{2}{x+3} = -\infty$

$\longrightarrow \lim_{x\to -3^+} f(x) = \lim_{x\to -3^+} x^2 = (-3)^2 = 9$

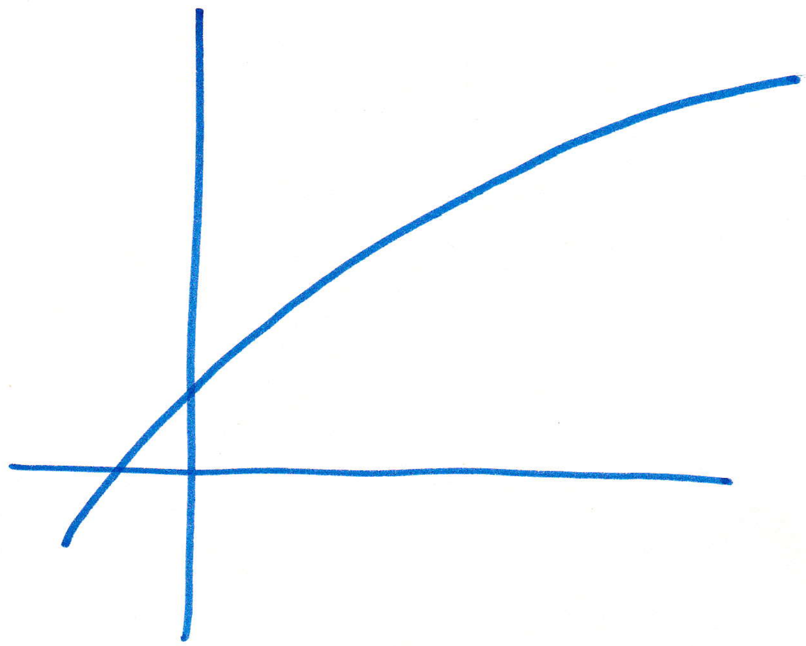
$\therefore \lim_{x\to -3} f(x)$ ไม่มีค่า

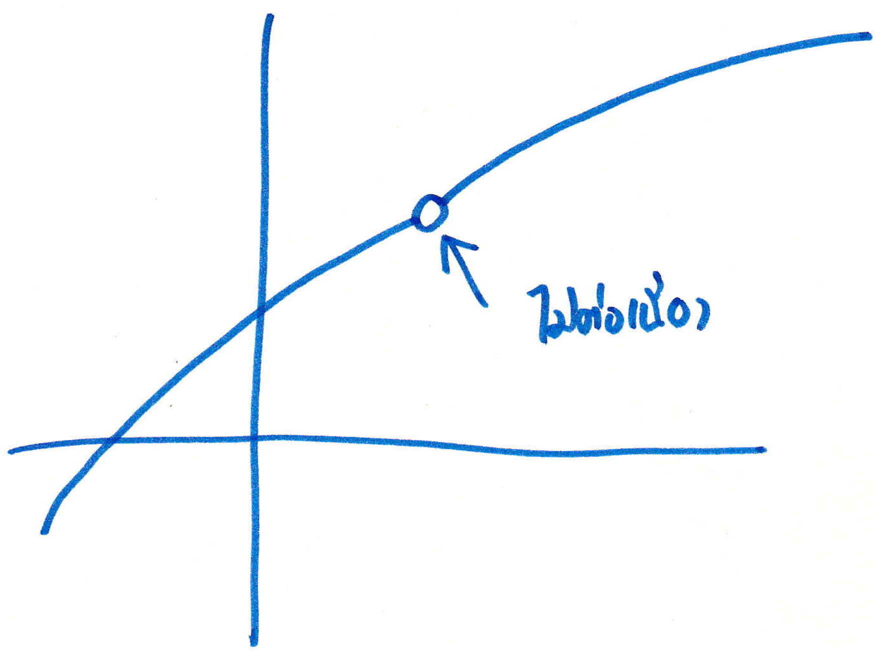
(b) $\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2 = 0$

(c) $\lim_{x\to 2} f(x) \longrightarrow \lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 = 4$

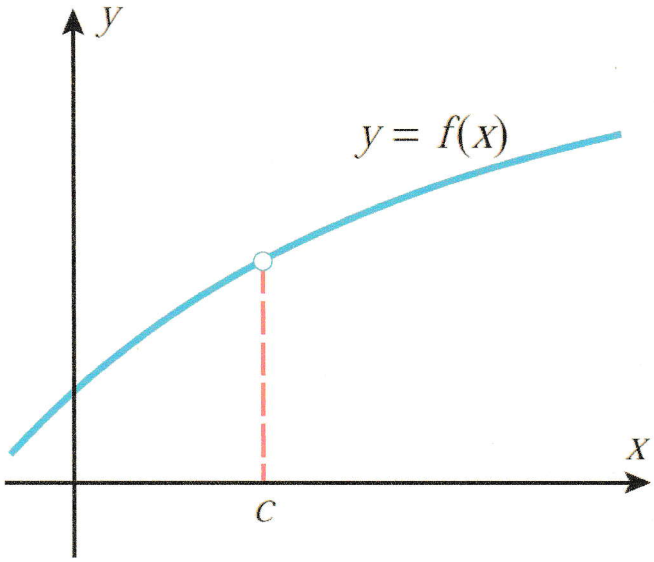
$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} 6-x = 4$

$\therefore \lim_{x\to 2} f(x) = 4$ #



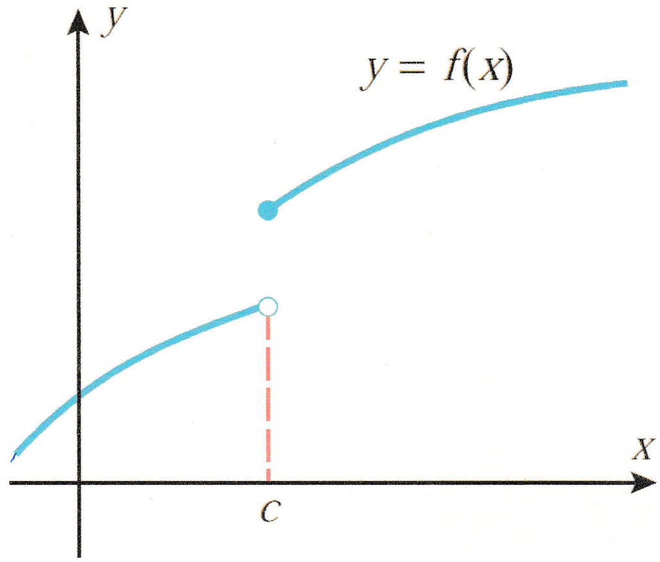

ไม่ต่อเนื่อง

Continuity



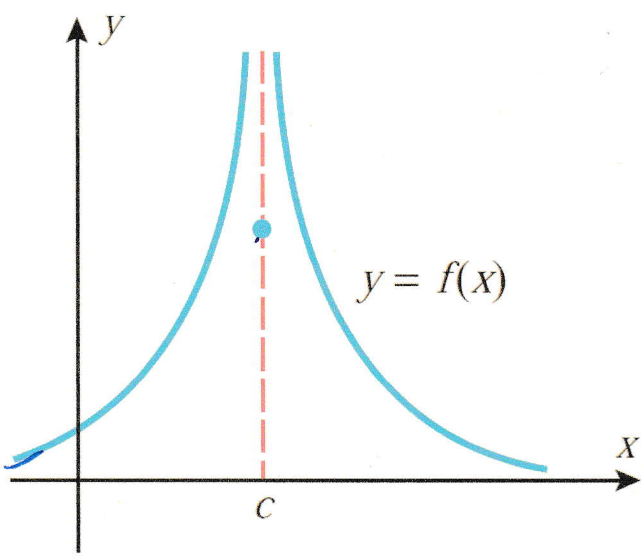
(a)

removable discontinuity



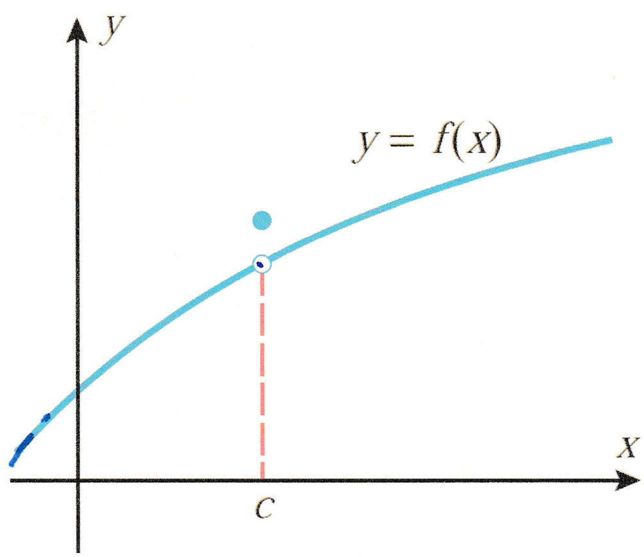
(b)

jump discontinuity



(c)

infinite discontinuity



(d)

removable discontinuity

DEFINITION A function f is said to be *continuous at $x = c$* provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

Example 14 Determine whether the following functions are continuous at $x = 2$.

$$f(x) = \frac{x^2 - 4}{x - 2}, \qquad g(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 2, & x = 2, \end{cases} \qquad h(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ 4, & x = 2. \end{cases}$$

① $f(2)$ ~~หาค่าไม่ได้~~ ไม่นิยาม $\therefore f$ ไม่ต่อเนื่องที่ $x = 2$

② (1) $g(2) = 2$

(2) $\lim_{x \to 2} g(x) = \lim_{x \to 2} \frac{x^2-4}{x-2} = \lim_{x \to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = 4$

(3) $g(2) \neq \lim_{x \to 2} g(x)$

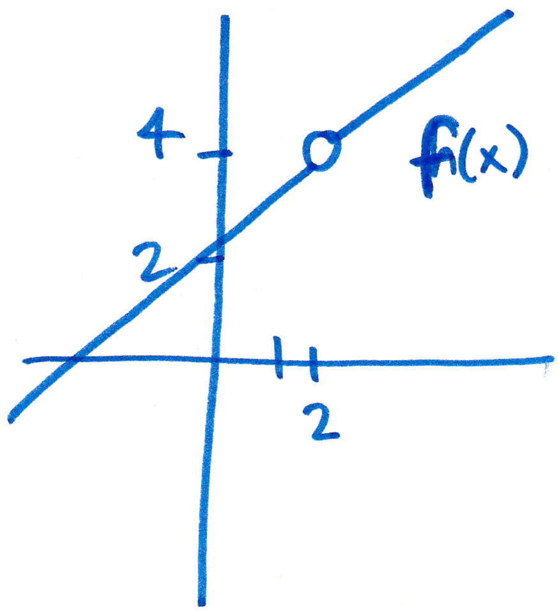
$\therefore g$ ไม่ต่อเนื่องที่ $x = 2$

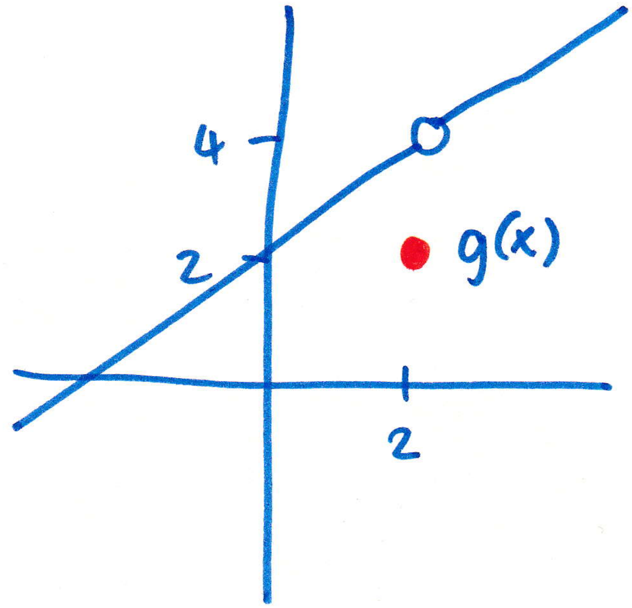
③ (1) $h(2) = 4$

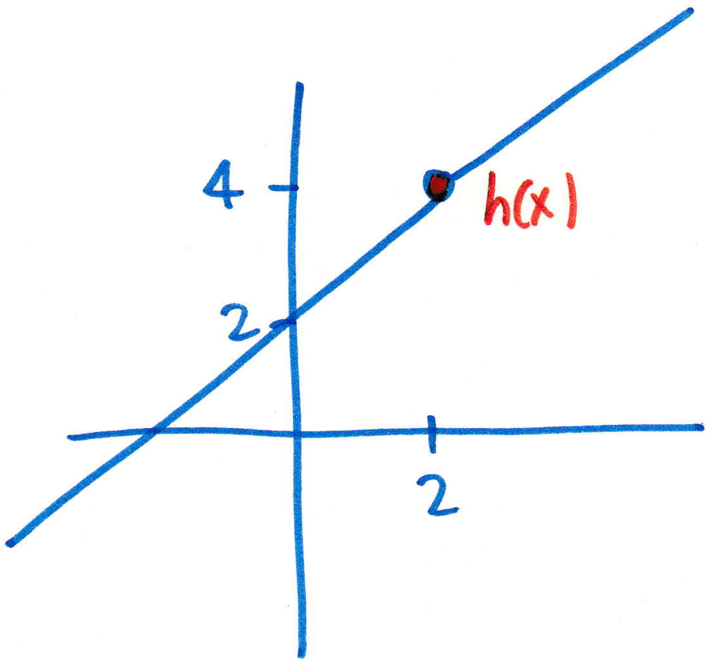
(2) $\lim_{x \to 2} h(x) = \lim_{x \to 2} \frac{x^2-4}{x-2} = 4$

(3) $h(2) = \lim_{x \to 2} h(x)$

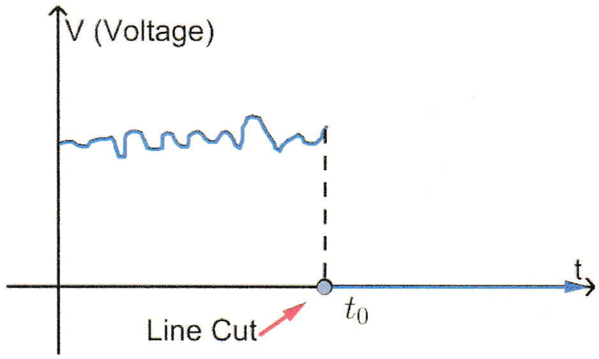
$\therefore h$ ต่อเนื่องที่ $x = 2$

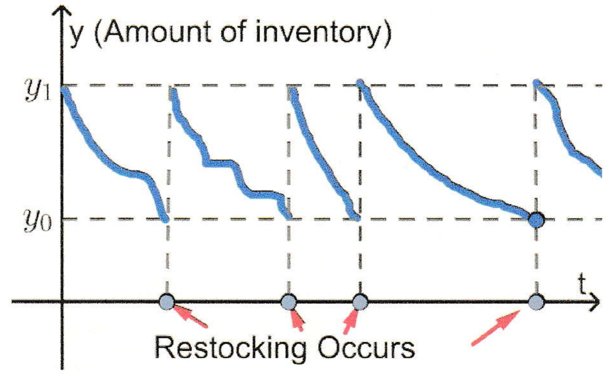

4
2
2
f(x)


4
2
2
g(x)


4
2
2
h(x)

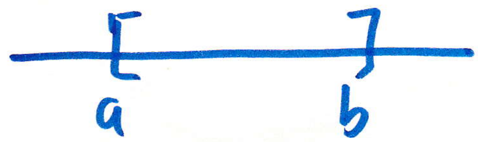
Continuity in Applications





1.3.1 Continuity on an Interval

We say a function f is *continuous from the left* at c if



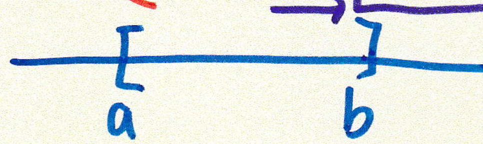
$$\lim_{x\to c^-} f(x) = f(c)$$

and is *continuous from the right* at c if

$$\lim_{x\to c^+} f(x) = f(c).$$

DEFINITION A function f is said to be *continuous on a closed interval* $[a, b]$ if the following conditions are satisfied:

$\boxed{f(a) = \lim_{x\to a^+} f(x)}$ $\boxed{f(b) = \lim_{x\to b^-} f(x)}$



1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.