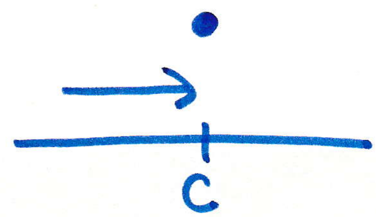
DEFINITION A function f is said to be *continuous at $x = c$* provided the following conditions are satisfied:

1. $f(c)$ is defined.
2. $\lim_{x \to c} f(x)$ exists.
3. $\lim_{x \to c} f(x) = f(c)$.

1.3.1 Continuity on an Interval

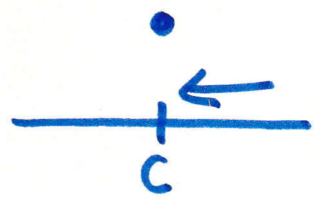
We say a function f is *continuous from the left* at c if

$$\lim_{x \to c^-} f(x) = f(c)$$



and is *continuous from the right* at c if

$$\lim_{x \to c^+} f(x) = f(c).$$



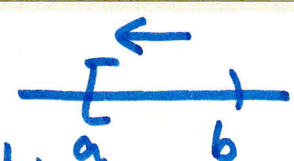
DEFINITION A function f is said to be *continuous on a closed interval* $[a, b]$ if the following conditions are satisfied:

1. f is continuous on (a, b).
2. f is continuous from the right at a.
3. f is continuous from the left at b.

$\lim_{x \to a^+} f(x) = f(a)$

$\lim_{x \to b^-} f(x) = f(b)$

f is continuous on $[a, b)$

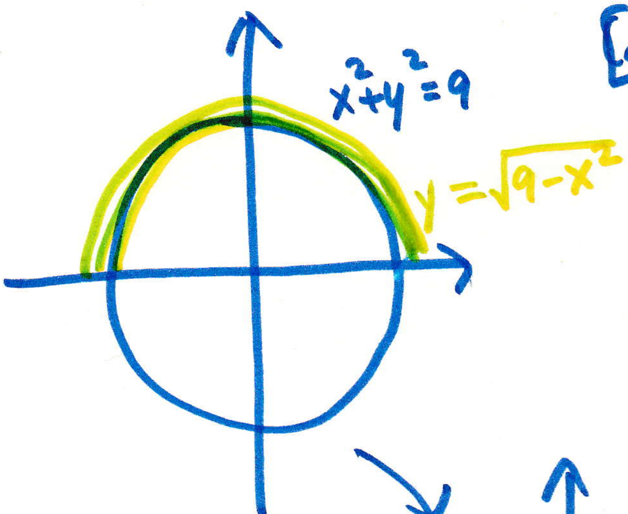


(1) f is continuous on (a, b)

(2) f is continuous from the right at a.

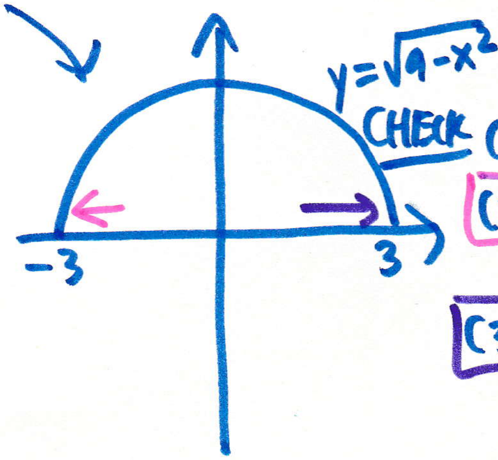
i.e. $\lim_{x \to a^+} f(x) = f(a)$

Example 15 Explain the continuity of the function $f(x) = \sqrt{9 - x^2}$ on the interval $[-3, 3]$.



[สมการวงกลม รัศมี r หน่วย จ.ศ.ก.(0,0)]

$$x^2+y^2=r^2$$
$$y^2=r^2-x^2$$
$$y=\pm\sqrt{r^2-x^2}$$



$D_f=[-3,3]$

CHECK (1) f is continuous on $(-3,3)$

$\boxed{(2)}$ f is continuous from the right at $x=-3$

$\boxed{(3)}$ f is continuous from the left at $x=3$

(1) สมมติให้ $c\in(-3,3)$ [CHECK ว่า f ต่อเนื่องที่ c ไหม]

(1) $f(c)=\sqrt{9-c^2}$

(2) $\lim_{x\to c} f(x)=\sqrt{9-c^2}$

(3) $\lim_{x\to c} f(x)=f(c)$ ทุก $c\in(-3,3)$

$\therefore f$ ต่อเนื่องบน $(-3,3)$

(2) CHECK $\lim_{x\to -3^+} f(x)=f(-3)$

(1) $f(-3)=0$

(2) $\lim_{x\to -3^+} f(x)=0$

(3) $\lim_{x\to -3^+} f(x)=f(-3)$

$\therefore f$ ต่อเนื่องทางขวาที่ $x=-3$

(3) CHECK $\lim_{x\to 3^-} f(x)=f(3)$

(1) $f(3)=0$

(2) $\lim_{x\to 3^-} f(x)=0$

(3) $\lim_{x\to 3^-} f(x)=f(3)$

$\therefore f$ ต่อเนื่องทางซ้ายที่ $x=3$

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

[ทุก $c \in \mathbb{R}$]

① $p(c)$ หาค่าได้

② $\lim_{x \to c} p(x)$ มีค่า

③ $\lim_{x \to c} p(x) = p(c)$

---

① $p(c) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_1 c + a_0$

② $\lim_{x \to c} p(x) = a_n c^n + a_{n-1} c^{n-1} + \dots + a_1 c + a_0$

③ $\lim_{x \to c} p(x) = p(c)$

$\therefore$ $p(x)$ ต่อเนื่องทุกจุดในโดเมนของ f

Rational function

(Rational Number)

จำนวนตรรกยะ:

Some Properties of Continuous Functions

THEOREM 1.5 If the functions f and g are continuous at c, then

(a) $f + g$ is continuous at c.

(b) $f - g$ is continuous at c.

(c) fg is continuous at c.

(d) f/g is continuous at c if $g(c) \neq 0$ and has a discontinuity at c if $g(c) = 0$.

Continuity of Polynomials and Rational Functions

THEOREM 1.6 ตัวเศษ : numerator

(a) A polynomial is continuous everywhere. WHY?

(b) A rational function is continuous at every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero. ตัวส่วน

Example 16 For what values of x is there a discontinuity in the graph of

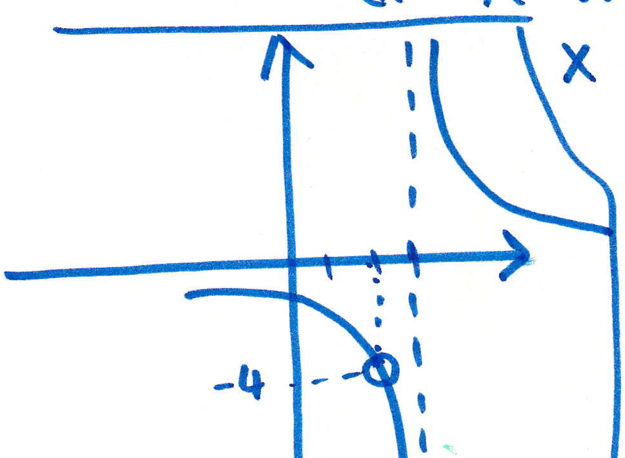
$$y = \frac{x^2 - 4}{x^2 - 5x + 6}?$$

By Thm. 1.6, we find x such that

$$x^2 - 5x + 6 = 0$$

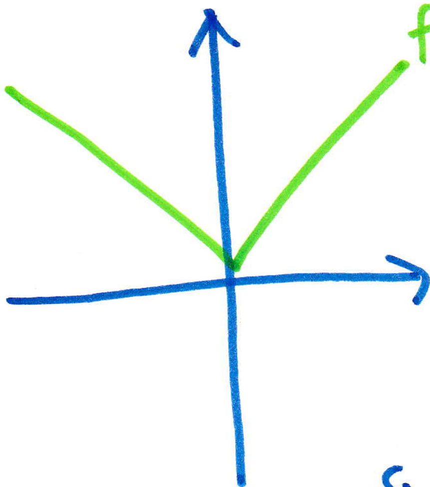
$$(x-2)(x-3) = 0$$

$$x = 2, 3$$



$$y = \frac{x^2-4}{x^2-5x+6} = \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}(x-3)} \quad ; x \neq 2$$

Example 17 Show that $|x|$ is continuous everywhere.



$f(x)=|x| \longrightarrow f(x)=\begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$

If $x>0$, $f(x)=x$
From Thm 1.6, we ~~have~~ obtain that $f(x)$ is continuous on $(0,\infty)$

Similarly, if $x<0$, $f(x)=-x$
We remark that $f(x)=-x$ is continuous on $(-\infty,0)$

$\therefore$ We consider at $x=0$.
CHECK continuity at $x=0$

(1) $f(0)$
(2) $\lim_{x\to 0} f(x)$
(3) $\lim_{x\to 0} f(x)=f(0)$

---

(1) $f(0)=0$
(2) $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$
$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x = 0$
$\because \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x)$, we have $\lim_{x\to 0} f(x)=0$
(3) $\lim_{x\to 0} f(x) = f(0)$
$\therefore$ f is continuous at $x=0$

$\therefore f(x)=|x|$ is continuous everywhere.

1.3.2 Continuity of Compositions

THEOREM 1.7 If $\lim_{x\to c} g(x) = L$ and if the function f is continuous at L, then $\lim_{x\to c} f(g(x)) = f(L)$. That is,

$$\lim_{x\to c} g(x) = L$$

$$\lim_{x\to c} f(g(x)) = f(\lim_{x\to c} g(x)).$$

Example 18 Given that $\lim_{x\to 2} x^2 - 9 = -5$ find, $\lim_{x\to 2} |x^2 - 9|$.

$f(x) = |x^2-9|$

คือ
$$f(x) = \begin{cases} x^2-9 & ; (x^2-9) \geq 0 \to (x-3)(x+3) \geq 0 \\ -(x^2-9) & ; \boxed{(x^2-9) < 0} \end{cases}$$

$+ \quad - \quad +$; -3, 3

$$f(x) = \begin{cases} x^2-9 & ; x \in (-\infty,-3] \cup [3,\infty) \\ -(x^2-9) & ; x \in (-3,3) \end{cases}$$

แล $\lim_{x\to 2} f(x) = \lim_{x\to 2} -(x^2-9) = 5$

มองว่า $f(x) = |x^2-9|$ คือฟังก์ชันของ $h(y) = |y| \leftarrow h(y)$ ต่อเนื่องทุกจุดใน จ.น.จริง

$y(x) = x^2-9$

$f(x) = h(y(x))$

จาก $h(y) = |y|$ ต่อเนื่องทุกจุดในจำนวนจริง

$$\lim_{x\to 2} |x^2-9| = \left|\lim_{x\to 2}(x^2-9)\right| = |-5| = 5.$$

THEOREM 1.8

(a) If the function g is continuous at c, and the function f is continuous at $g(c)$, then the composition $f \circ g$ is continuous at c.

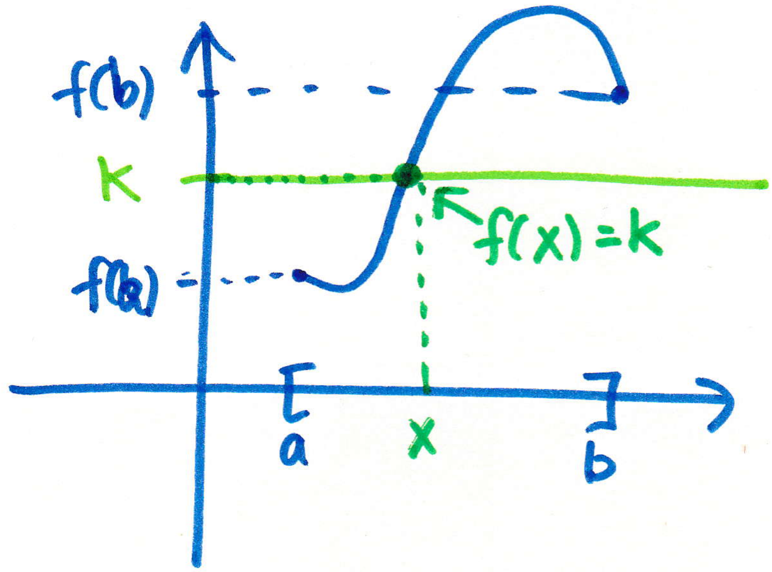
(b) If the function g is continuous everywhere, and the function f is continuous everywhere, then the composition $f \circ g$ is continuous everywhere.

Theorem

Trigonometric functions, inverse trigonometric functions, exponential functions, logarithmic functions are continuous in on their domains.

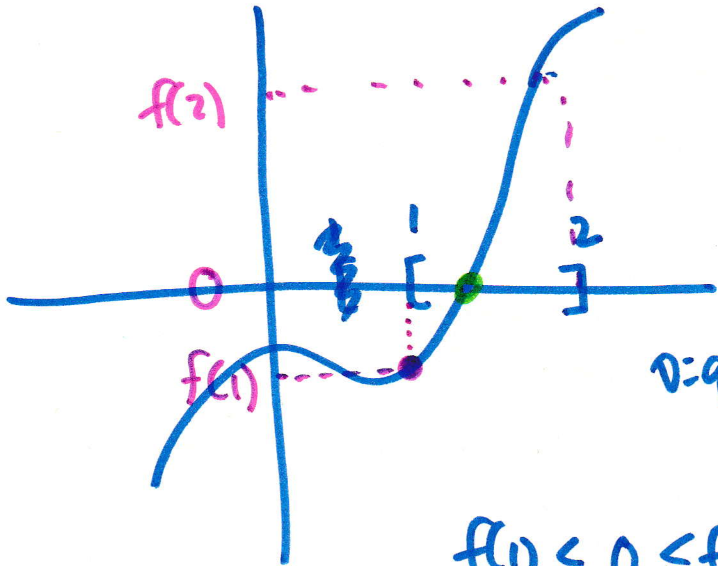
1.4 The Intermediate-Value Theorem

THEOREM 1.9 (INTERMEDIATE-VALUE THEOREM) If f is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, inclusive, then there is at least one number x in the interval $[a, b]$ such that $f(x) = k$. $f(a) \le k \le f(b)$



$p(x) = x^3 - x - 1$

$f(x) = x^3 - x - 1 = 0$



<u>Root of equation</u> :

จุดตัดระหว่าง $f(x)$ กับ $y=0$

ถ้า $y=0$ แล้ว $x = ?$

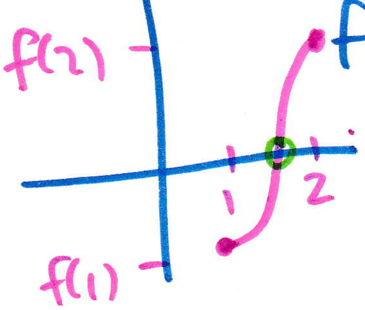
$f(x)=x^3-x-1$ ต่อเนื่องบน $[1,2]$

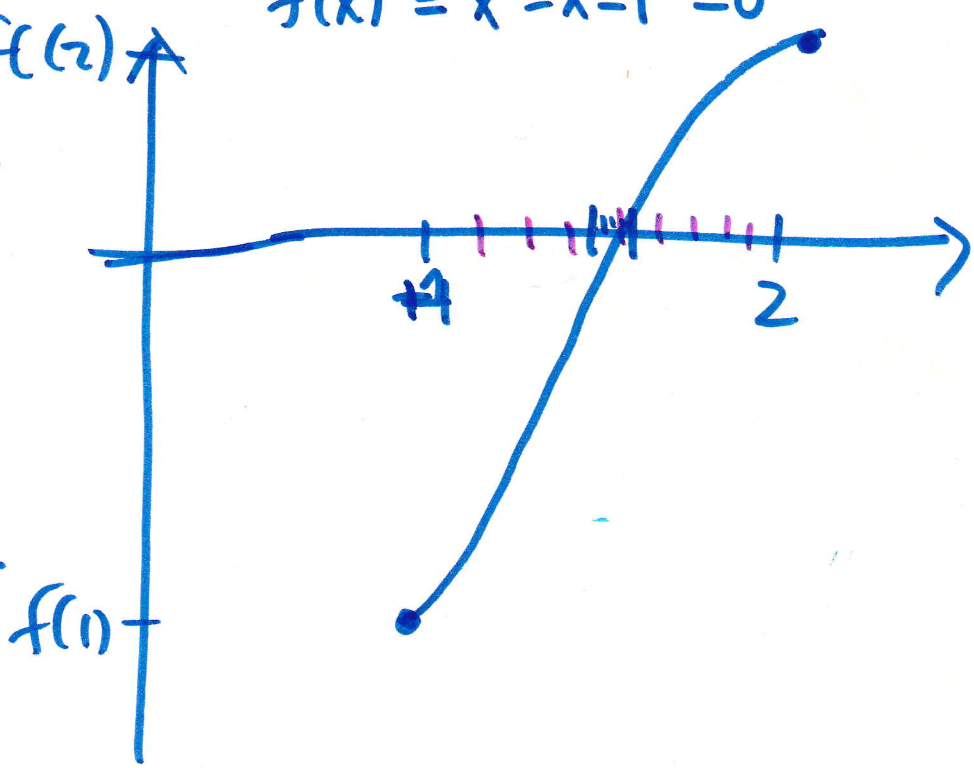
จะดูว่า สมการตัดกันระหว่าง f กับ $y=0$ หรือไม่

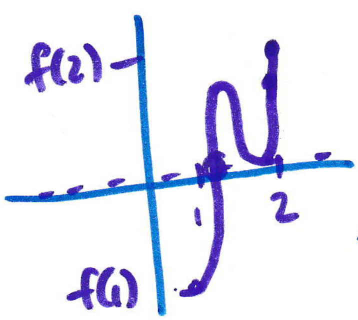
$f(1) \leq 0 \leq f(2)$

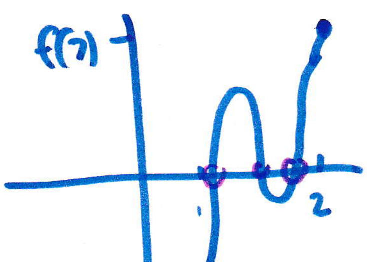
จึงโดย <u>IVP</u> ได้ว่า มี $x \in [1,2]$ ที่ทำให้

$f(x) = x^3 - x - 1 = 0$









Example 19 Verify that there exists at least one root of the equation $x^3 - x - 1 = 0$ in the closed interval $[1, 2]$. Then, approximate this root to two decimal-place accuracy.

$f(x) = x^3 - x - 1$

Since $f(1) = -1$ and $f(2) = 5$, we have $f(1) \le 0 \le f(2)$. Therefore the root is between 1 and 2.

x	1	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2
$f(x)$	−1	−0.76	−0.47	−0.10	0.34	0.87	1.49	2.21	3.09	3.79	5

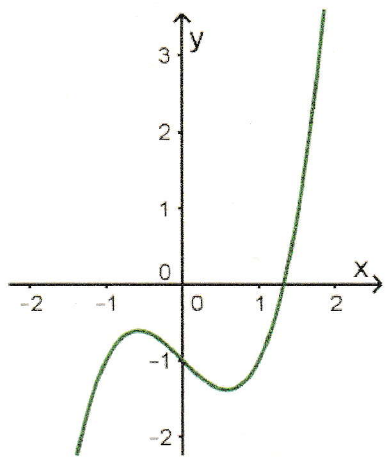


Figure 1.13: Graph of $y = x^3 - x - 1$

Since $f($1.3$) < 0$ and $f($1.4$) > 0$, the root is between 1.3 and 1.4.

x	1.30	1.31	1.32	1.33	1.34	1.35
$f(x)$	−0.103	−0.061	−0.020	0.023	0.066	0.110
x	1.36	1.37	1.38	1.39	1.40	
$f(x)$	0.155	0.201	0.248	0.296	0.344	

Since $f($1.32$) < 0$ and $f($1.33$) > 0$, the root is between 1.32 and 1.33.

x	1.320	1.321	1.322	1.323	1.324	1.325
$f(x)$	−0.020	−0.015	−0.011	−0.007	−0.0031	0.0012
x	1.326	1.327	1.328	1.329	1.330	
$f(x)$	0.0055	0.0098	0.0140	0.0183	0.0226	

The root of the equation $x^3 - x - 1 = 0$, that is between 1 and 2, is approximately $x \approx 1.32$.

IVP ทฤษฎีบทค่าระหว่างกลาง

$x^3 - x - 1 = 5$ บนช่วง $[-10, 10]$

ลอง $f(x) = x^3 - x - 1$

$f(-10) = (-1000 + 10 - 1) = -991$

$f(10) = 1000 - 10 - 1 = 989$

$f(-10) \le 5 \le f(10)$ $\quad \therefore$ มี $x \in [-10, 10]$ ที่ทำให้

$\boxed{x^3 - x - 1 = 5}$