ทบทวน

- Limit
- Continuity

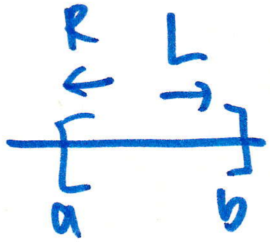
f continuity at $x = ♡$

(1) $f(♡)$ exists

(2) $\lim_{x \to ♡} f(x)$ exists

(3) $\lim_{x \to ♡} f(x) = f(♡)$

f continuity on $[a,b]$

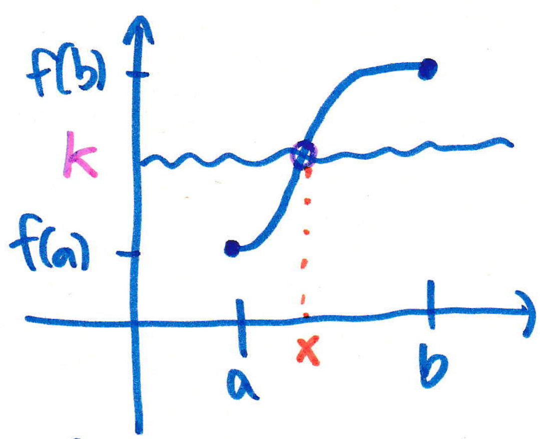


(1) f is continuous on (a,b)

(2) f is continuous from the right at $x=a$

(3) f is continuous from the left at $x=b$

⊛ IVP : Intermediate-value Theorem



f cont, $f(a) \leq k \leq f(b)$, $\Rightarrow \exists x \in [a,b]$ such that $f(x) = k$

<u>Derivative of function</u>

Revision: Definition of Derivative

First Approach

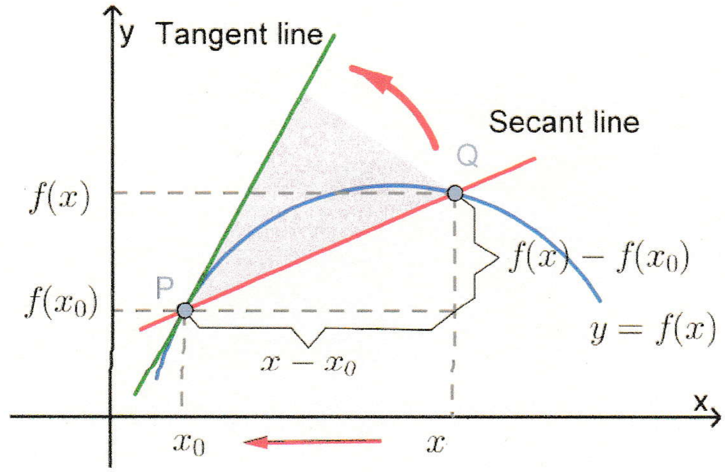


Figure 2.1: The slope of the tangent line as a limit of slopes of secant lines

$$\boxed{m_{tan} = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}}$$

Second Approach

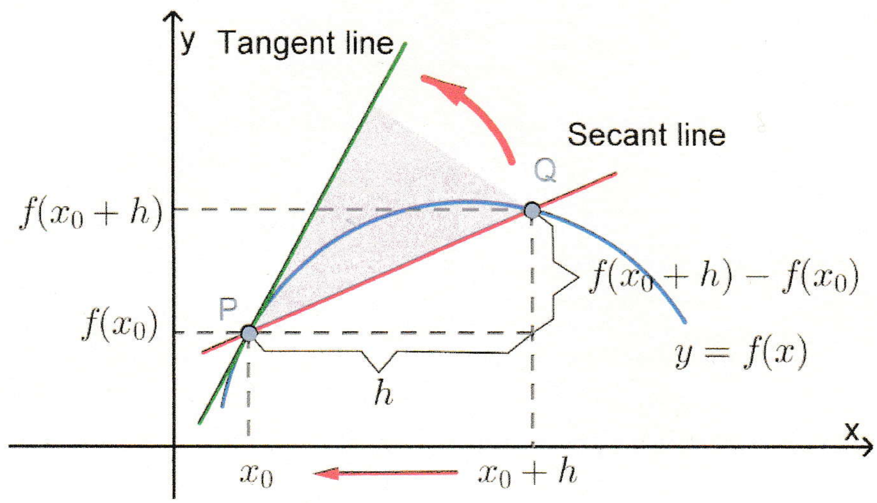
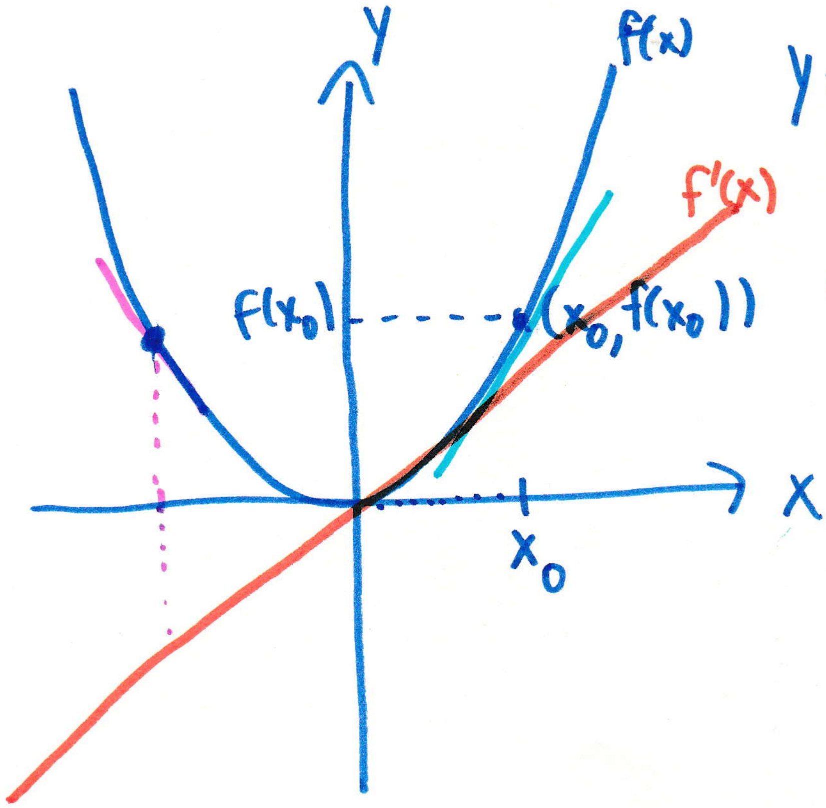


Figure 2.2: The slope of the tangent line as a limit of slopes of secant lines

$$\boxed{m_{tan} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}}$$

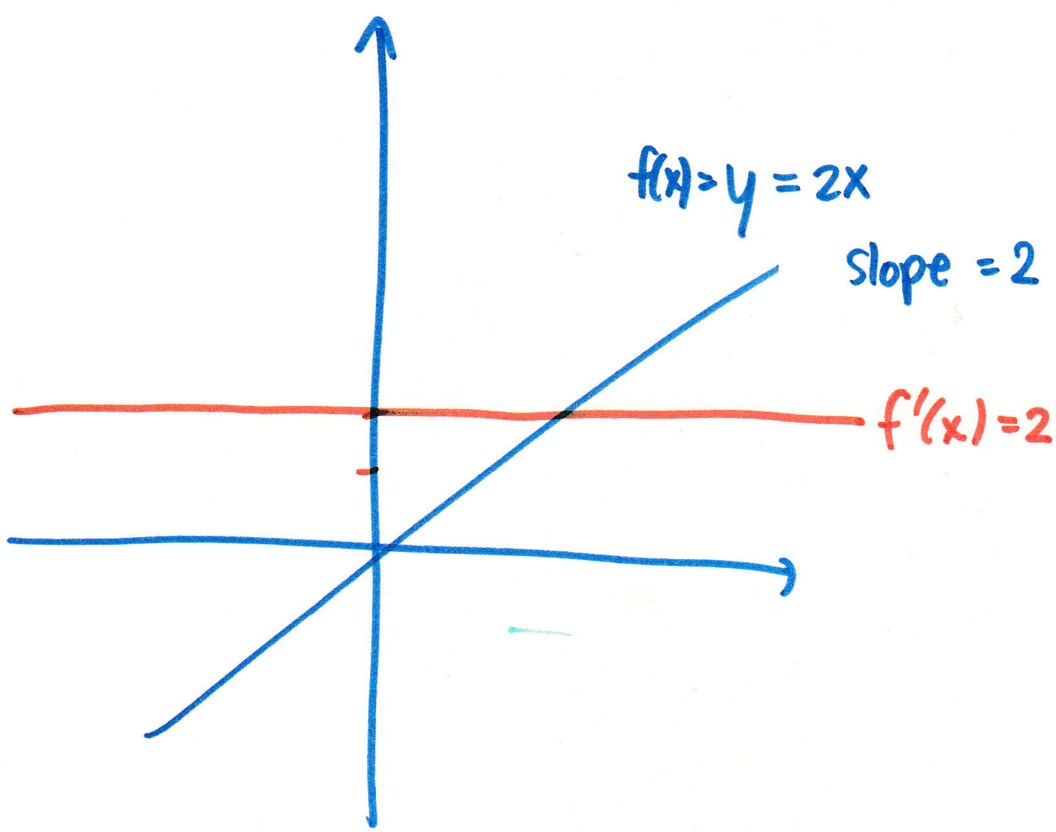
$$\boxed{f'(x) = \lim \frac{f(x+h) - f(x)}{h}}$$ ← อนุพันธ์ของฟังก์ชัน



$y = x^2$

$f(x) = x^2$

อนุพันธ์ $f'(x) = 2x$



Example 7 Find the derivative with respect to x of $f(x) = x^3 - x$.

$$(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{[(x+h)^3 - (x+h)] - (x^3 - x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^3} + 3x^2h + 3xh^2 + h^3 - \cancel{x} - h - \cancel{x^3} + \cancel{x}}{h}$$

$$= \lim_{h \to 0} 3x^2 + 3xh + h^2 - 1$$

$$f'(x) = 3x^2 - 1$$

หา อนุพันธ์ของฟังก์ชัน ที่ $x = 1$

$f(1) = 0$	$f(-1) = 0$
$f'(1) = 2$	$f'(-1) = 2$
	$f(0) = 0$
	$f'(0) = -1$

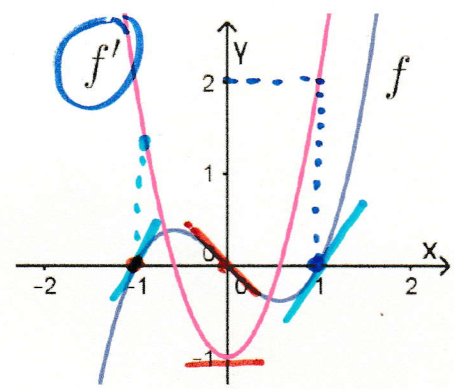
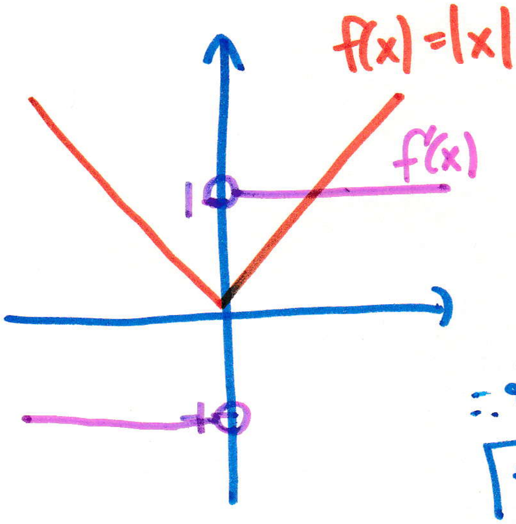


Figure 2.3: Graph of f and f' in Example 7

If we graph f and f' together as in Figure 2.3, we can see the relationship between them. Since $f'(x)$ is the slope of the tangent line to the graph of $y = f(x)$ at x, it follows that f' is positive, negative, and zero where the tangent line has positive slope, has negative slope, and is horizontal, respectively.

Example 8 (a) Prove that $f(x) = |x|$ is not differentiable at $x = 0$ by considering the limit.

(b) Find a formula for $f'(x)$.



$$f(x) = \begin{cases} x & \text{where } x \geq 0 \\ -x & \text{where } x < 0 \end{cases}$$

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

∴ ที่ $x = 0$

$\boxed{\text{จะเช็ค}}$ $f'(0) = \boxed{\lim_{h \to 0} \frac{f(0+h) - f(0)}{h}}$ มีไหม

พิจารณาลิมิตซ้ายและลิมิตขวา

$$\lim_{h \to 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^-} \frac{-(0+h) - 0}{h} = -1$$

$$\lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{(0+h) - 0}{h} = 1$$

$\therefore \lim_{h \to 0} \frac{f(0+h) - f(0)}{h}$ ไม่มีค่า

$\therefore f'(0)$ ไม่นิยาม

↓

$f(x) = |x|$ is not differentiable at $x = 0$.

$$f'(x) = \begin{cases} 1 & \text{where } x > 0 \\ -1 & \text{where } x < 0 \end{cases}$$

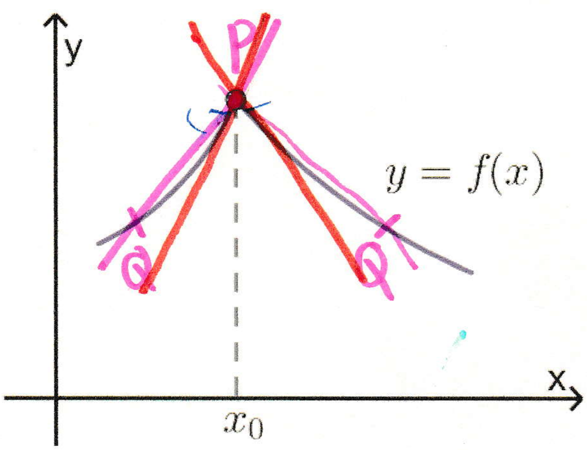
2.2.2 Differentiability การหาอนุพันธ์ได้

Definition A function f is ***differentiable at*** x_0 if the limit

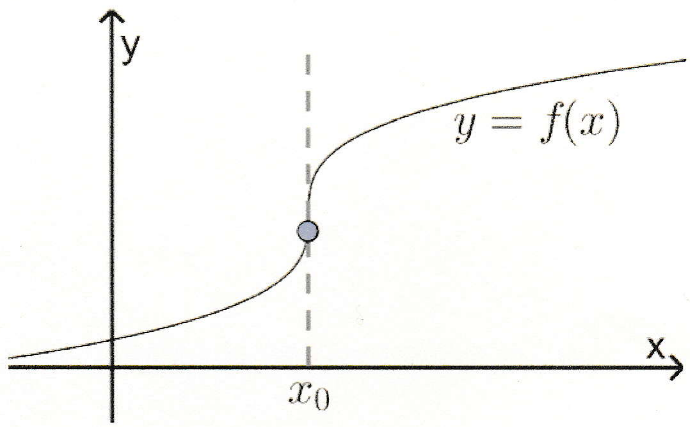
$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$ ⇒ ถ้าลิมิตหาค่าได้

exists. If f is differentiable at each point of the open interval (a, b), then we say that it is ***differentiable on*** (a, b) , and similarly for open intervals of the form $(a, +\infty)$, $(-\infty, b)$, and $(-\infty, +\infty)$. In the last case, we say that f is ***differentiable everywhere*** .

Figure 2.4 illustrates two common ways in which a function that is continuous at x_0 is not differentiable at x_0. Since the slopes of the secant lines have different limits from the left and from the right.

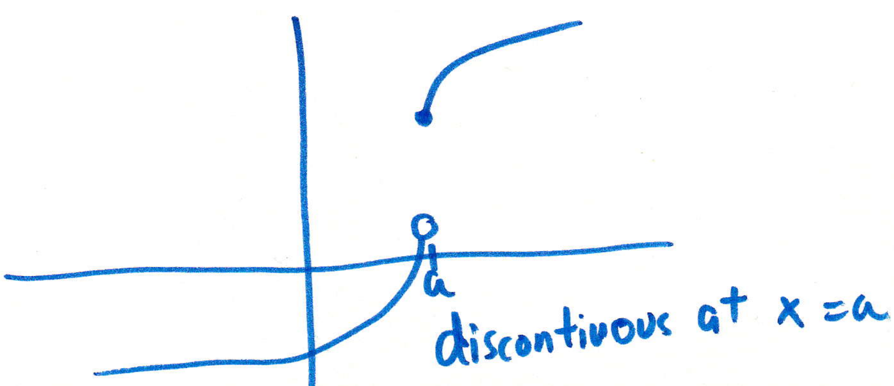


Corner point

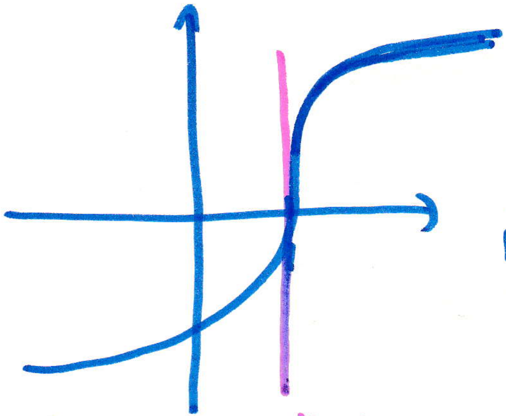


Point of vertical tangency

Figure 2.4: Continuous at x_0 but not differentiable at x_0



let $f(x) = (x-1)^{\frac{1}{3}}$, Is f differentiable at $x = 1$?



check ว่า $\lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$ มีค่าไหม

$$\lim_{h\to 0} \frac{f(1+h)-f(1)}{h} = \lim_{h\to 0} \frac{(1+h-1)^{\frac{1}{3}} - (1-1)^{\frac{1}{3}}}{h}$$

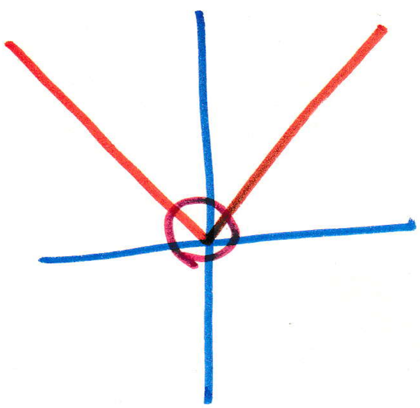
$$= \lim_{h\to 0} \frac{h^{\frac{1}{3}} - 0}{h}$$

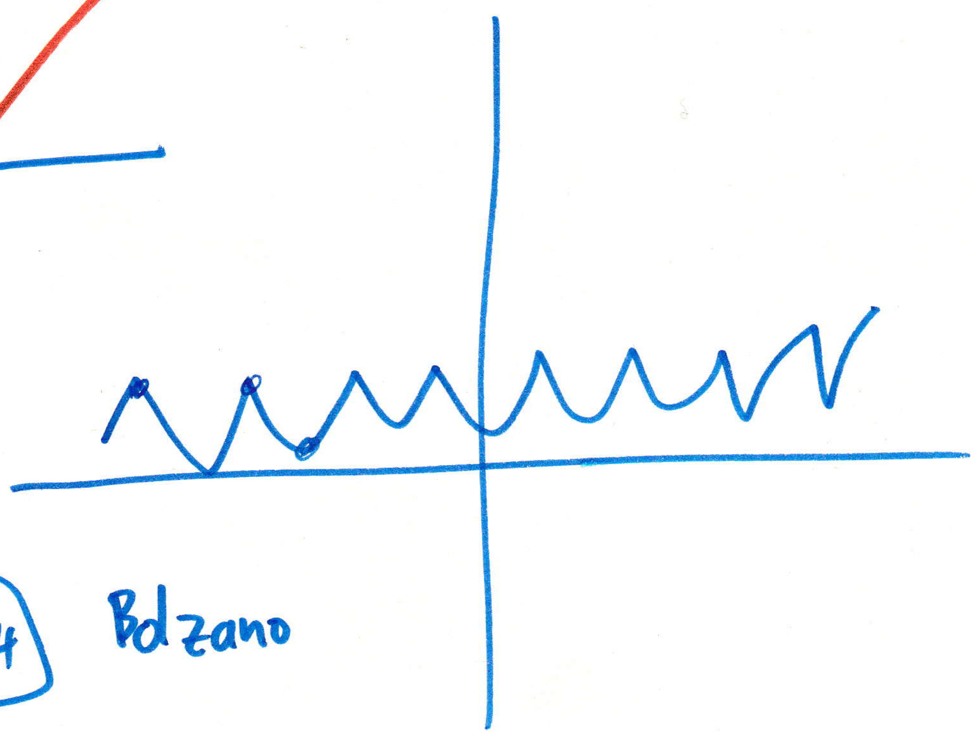
$$= \lim_{h\to 0} \frac{1}{h^{2/3}}$$

$$= +\infty$$

จะเห็นว่า $\lim\limits_{h\to 0} \dfrac{f(1+h)-f(1)}{h}$ หาค่าไม่ได้ $(+\infty)$

$\therefore f'(1)$ ไม่มี





$\boxed{1834}$ Bolzano

A function which is continuous **everywhere** but differentiable **nowhere**: **Weierstrass function** (1860)

$$f(x) = \sum_{n=0}^{\infty} a^n \cos(b^n \pi x)$$

where $0 < a < 1$, b is a positive odd integer, and

$$ab > 1 + \frac{3}{2}\pi.$$

Karl Weierstrass, "Über continuirliche Functionen eines reellen Arguments, die für keinen Werth des letzeren einen bestimmten Differentialquotienten besitzen," (On continuous functions of a real argument which possess a definite derivative for no value of the argument) in: Königlich Preussichen Akademie der Wissenschaften, *Mathematische Werke von Karl Weierstrass* (Berlin, Germany: Mayer & Mueller, 1895), vol. 2, pages 71–74.

The relationship Between Differentiability and Continuity

$p \Rightarrow q \equiv \sim q \Rightarrow \sim p$

ถ้า f Diff ได้ที่ x_0 แล้ว f ต่อเนื่องที่ x_0

Theorem 2.1 If a function $\underbrace{f \text{ is differentiable at } x_0}_{p}$, then $\underbrace{\boxed{f \text{ is continuous at } x_0}}_{q}$.

Note that the converse of the theorem above is false, since continuous does not imply differentiable.

∴ If f is NOT continuous at x_0,
then f is NOT differentiable at x_0.

2.2.3 Other Derivative Notations

$y = f(x) \Rightarrow f'(x)$, y', $y'(x)$

When the independent variable is x, the derivative is commonly denoted by

$$f'(x) \quad \text{or} \quad \frac{d}{dx}[f(x)] \quad \text{or} \quad D_x[f(x)]$$

In the case where the dependent variable is $y = f(x)$, the derivative is also denoted by

$$y' \quad \text{or} \quad y'(x) \quad \text{or} \quad \frac{dy}{dx}$$

With the above notations, the value of the derivative at a point x_0 can be expressed as

$$f'(x_0) \quad \text{or} \quad \frac{d}{dx}[f(x)]\Big|_{x=x_0} \quad \text{or} \quad D_x[f(x)]|_{x=x_0} \quad \text{or} \quad y'(x_0) \quad \text{or} \quad \frac{dy}{dx}\Big|_{x=x_0}$$

evaluating symbol
สัญลักษณ์การแทนค่า

Derivative as the rate of change

Rectilinear Motions

The position coordinate of a particle in rectilinear motion at time t is $s = f(t)$. The *average velocity* of the particle over a time interval $[t_0, t_0 + h]$ for $h > 0$ is

$$v_{ave} = \frac{\text{change in position}}{\text{time elapsed}} = \frac{f(t_0 + h) - f(t_0)}{h}.$$

Example 4 Suppose that $s = f(t) = 1 + 5t - 2t^2$ is the position function of a particle, where s is in meters and t is in seconds. Find the average velocities of the particle over the time intervals (a) $[0, 2]$ and (b) $[2, 3]$.

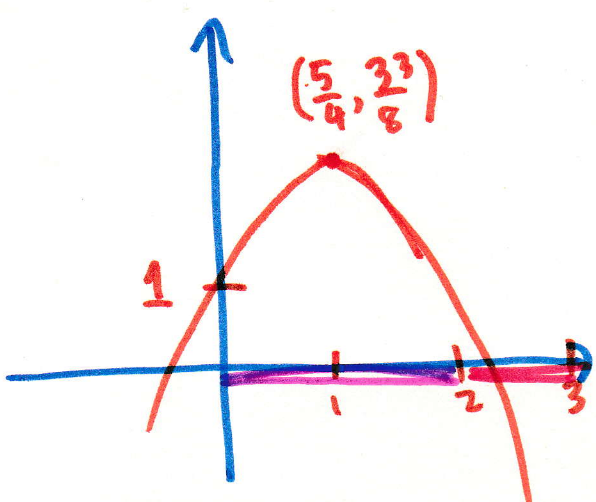
$$f(t) = 1 + 5t - 2t^2 = -2\left(t - \frac{5}{4}\right)^2 + \frac{33}{8}$$

$y = ax^2 + bx + c$

$\boxed{y = a(x-h)^2 + k}$

$$v_{\text{average velocity in } [0,2]} = \frac{f(2) - f(0)}{2 - 0} = \frac{(1+10-8) - 1}{2} = 1 \text{ m/s}$$

$$v_{\text{average velocity in } [2,3]} = \frac{f(3) - f(2)}{3 - 2} = \frac{(1+15-18) - (1+10-8)}{1} = \frac{-2-3}{1} = -5 \text{ m/s}$$



Taking the limit of v_{ave} as $h \to 0$, we get instantaneous velocity v_{inst} of the particle at time t_0.

$$v_{inst} = \lim_{h \to 0} \frac{f(t_0+h) - f(t_0)}{h}.$$

Example 5 Consider particle in Example 4, where

$$s = f(t) = 1 + 5t - 2t^2.$$

The position of the particle at time $t = 2$ s is $s = 3$ m. Find the instantaneous velocity of the particle at time $t = 2$ s.

$$\begin{aligned}
v_{inst}(2) &= \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h \to 0} \frac{[1 + 5(2+h) - 2(2+h)^2] - 3}{h} \\
&= \lim_{h \to 0} \frac{[1 + 10 + 5h - 2(4 + 4h + h^2)] - 3}{h} \\
&= \lim_{h \to 0} \frac{[\cancel{11} + 5h - \cancel{8} - 8h - 2h^2 - \cancel{3}]}{h} \\
&= \lim_{h \to 0} \frac{-3\cancel{h} - 2h^2}{\cancel{h}} \\
&= -3 \quad \text{m/s}
\end{aligned}$$

ความเร็วในทิศทางตรงกันข้าม

ถ้า $s = f(t)$ เป็นฟังก์ชันของระยะทาง แล้ว

s' เป็นฟังก์ชันของความเร็ว $(v(t))$

$\therefore$ v' เป็นฟังก์ชันของความเร่ง $(a(t))$