Differentiability

Let f be a function

$$\boxed{f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}}$$

CASE

① $\displaystyle \lim_{h \to 0^-} \frac{f(x_0+h)-f(x_0)}{h} \neq \lim_{h \to 0^+} \frac{f(x_0+h)-f(x_0)}{h}$

② $\displaystyle \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} = +\infty$ เส้นสัมผัสแนวยืน

Thm 2.1 ถ้า f diff ได้ แล้ว f ต่อเนื่อง

$\equiv$ ถ้า f ไม่ต่อเนื่อง แล้ว f diff ไม่ได้

Ex ในQuiz $f(x) = |x|$ ต่อเนื่อง แต่ไม่สามารถหาอนุพันธ์ได้ที่ $x = 1$

$$f(x) = \begin{cases} x \; ; & x \geq 0 \\ -x \; ; & x < 0 \end{cases}$$

พิจารณา $f'(x)$ จาก $\displaystyle f'(x) = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h}$ $\nearrow \displaystyle \lim_{h \to 0^+} \frac{f(x+h)-f(x)}{h}$ $\searrow \displaystyle \lim_{h \to 0^-} \frac{f(x+h)-f(x)}{h}$

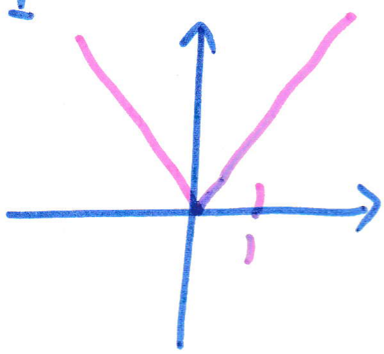
พิจารณา $x_0 = 1$

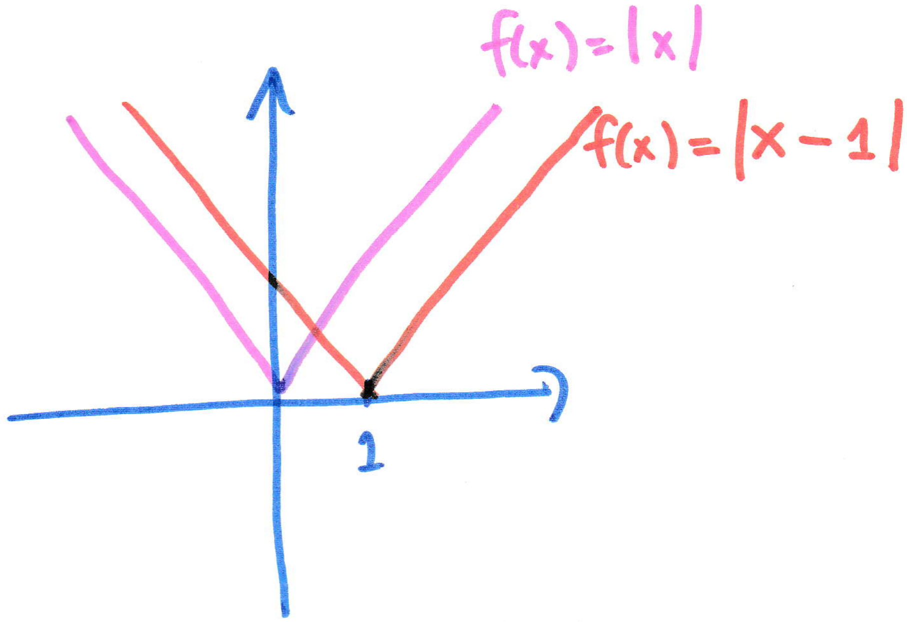
$$\lim_{h \to 0^+} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^+} \frac{(1+h)-1}{h} = 1$$

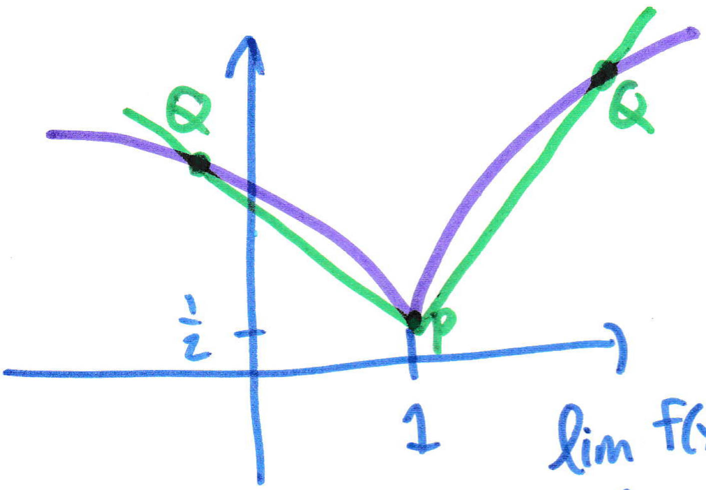
$$\lim_{h \to 0^-} \frac{f(1+h)-f(1)}{h} = \lim_{h \to 0^-} \frac{(1+h)-1}{h} = 1$$

$$\therefore \lim_{h \to 0} \frac{f(1+h)-f(1)}{h} = 1$$

$\therefore f$ หาอนุพันธ์ได้ที่ $x = 1$.

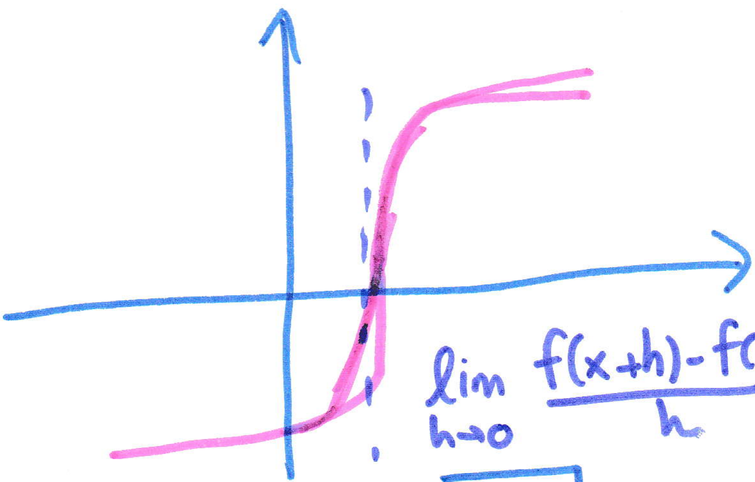






$$\lim_{x \to 1} f(x) = \frac{1}{2}$$

Case นี้ limit ของ ความชันของเส้นตัดโค้งไม่เท่ากัน.

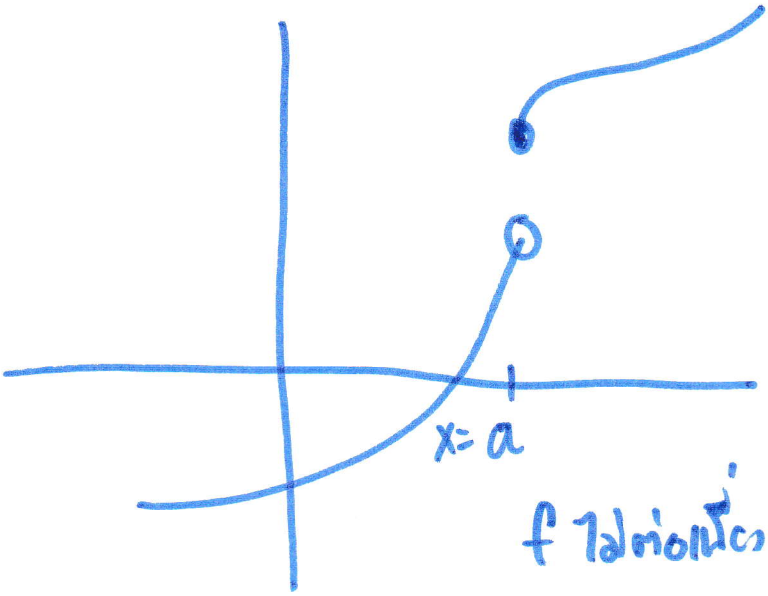


$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = +\infty$$

เส้นสัมผัสแนวยืน

* $x = a$ เป็นเส้นกำกับแนวยืน $\Longleftrightarrow \lim_{x \to a} f(x) = +\infty / -\infty$

หรือ $x \to a^{+}$ / $x \to a^{-}$



f ไม่ต่อเนื่อง $\Rightarrow$ f diff ไม่ได้

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f เป็นฟังก์ชันต่อเนื่องที่ $x = a$

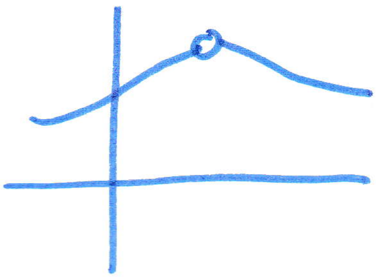
① $f(a)$ หาค่าได้

② $\lim_{x \to a} f(x)$ มีค่า

③ $\lim_{x \to a} f(x) = f(a)$

__ความผิดพลาด__

① เขียนเฉพาะ $\lim_{x \to a} f(x)$ มี



② f เป็น piecewise-defined f^n

2.3 Basic Differentiation Formulas

RECAP Henceforth, we will use the following notation for the derivative of a function $y = f(x)$.

- $f'(x)$
- $\frac{d}{dx}[f(x)]$
- $\frac{dy}{dx}$
- y'

Derivative of a constant

THEOREM 2.2 (THE CONSTANT RULE) Let c be a constant and $f(x) = c$ be a constant function, then

$$\frac{d}{dx}[c] = 0.$$

Proof

$$f(x) = c$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{c - c}{h}$$

$$= 0 \qquad \#$$

Example 9

$\frac{d}{dx}[5] = $ 0 $\quad \frac{d}{dx}[-7] = $ 0 $\quad \frac{d}{dx}[5^2] = $ 0

$\frac{d}{dx}[\pi] = $ 0 $\quad \frac{d}{dx}[\sqrt{2}] = $ 0 $\quad \frac{d}{dx}[\pi^2] = $ 0

Derivatives of power functions

Theorem 2.3 (The Power Rule) If n is any real number, then

$$\frac{d}{dx}[x^n] = nx^{n-1}.$$

$f(x) = x^n$

$\rightarrow f(x+h) = (x+h)^n$

Example 10

$= x^{-5} = -5x^{-5-1} = -5x^{-6} = -\frac{5}{x^6}$

$\frac{d}{dx}[x^5] = 5x^4$ $\quad \frac{d}{dx}[x] = 1$ $\quad \frac{d}{dx}[x^{12}] = 12x^{11}$

$\frac{d}{dx}[x^{-5}] = -5x^{-5-1} = -5x^{-6}$ $\quad \frac{d}{dx}[\frac{1}{x^5}] = -\frac{5}{x^6}$ $\quad \frac{d}{dx}[x^{\sqrt{2}}] = \sqrt{2}x^{\sqrt{2}-1}$

$\frac{d}{dx}[x^{\frac{2}{5}}] = \frac{2}{5}x^{-\frac{3}{5}}$ $\quad \frac{d}{dx}[\sqrt{x}] = \frac{1}{2\sqrt{x}}$ $\quad \frac{d}{dx}[\frac{1}{\sqrt{x}}] = \ldots\ldots$

$\frac{d}{dx}[x^{\frac{1}{2}}]$

$\frac{d}{dx}(x^{-\frac{1}{2}}) = -\frac{1}{2}x^{-\frac{1}{2}-1} = -\frac{1}{2}x^{-\frac{3}{2}}.$

Derivative of a constant times a fucntion

Theorem 2.4 (The Constant Multiple Rule) If f is differentiable at x and c is any real number, then cf is also differentiable at x and

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)].$$

Example 11 $\frac{d}{dx}[4x^8] = 4\frac{d}{dx}(x^8) = (4)(8x^7) = 32x^7$

$\frac{d}{dx}[-x^{12}] = -12x^{11}$

$\frac{d}{dx}[\frac{x}{\pi}] = \frac{1}{\pi}$

$(\frac{1}{\pi})(x)$

Derivatives of sums and diffences

THEOREM 2.5 (THE SUM AND DIFFERENCE RULES) If f and g are differentiable at x, then so are $f+g$ and $f-g$ and

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}[f(x)] \pm \frac{d}{dx}[g(x)].$$

Example 12 $\frac{d}{dx}[x^8 + x^2] =$ $8x^7 + 2x$

$\frac{d}{dx}[6x^{11} - 9] =$ $66x^{10}$

$\frac{d}{dx}[\frac{1}{x} - x^{-9}] =$ $-\frac{1}{x^2} + 9x^{-10} = -\frac{1}{x^2} + \frac{9}{x^{10}}$

$\frac{d}{dx}[\frac{\sqrt{x} - 3x}{\sqrt{x}}] =$ $\frac{d}{dx}[1 - 3\sqrt{x}] = 0 - 3\frac{d}{dx}[\sqrt{x}]$ $(x^{\frac{1}{2}})$

$= (-3)(\frac{1}{2}x^{-\frac{1}{2}})$

$= -\frac{3}{2\sqrt{x}}$

2.4 The Product and Quotient Rules

2.4.1 Derivative of a Product

Theorem 2.6 (The Product Rule) If f and g are differentiable at x, then so is the product $f \cdot g$, and

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$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)].$$

Example 16 1. Let $y = \underbrace{(4x^2 - 1)}_{f(x)}\underbrace{(7x^3 + x)}_{g(x)}$. Find dy/dx.

Solution.

$$y' = (4x^2-1)\frac{d}{dx}(7x^3+x) + (7x^3+x)\cdot\frac{d}{dx}(4x^2-1)$$

$$f(x) \cdot g'(x) + g(x) \cdot f'(x)$$

$$= (4x^2-1)(21x^2+1) + (7x^3+x)(8x)$$

$$= 140x^4 - 9x^2 - 1$$

$$y = (4x^2-1)(7x^3+x) = 28x^5 + 4x^3 - 7x^3 - x$$

$$= 28x^5 - 3x^3 - x$$

$$y' = 140x^4 - 9x^2 - 1$$

\#

2. Let $f(x) = \underbrace{(3x + 1)}_{f(x)}\underbrace{(2x^2 + 5)}_{g(x)}$. Find $f'(0)$.

Solution.

$$f'(x) = (3x+1)\frac{d}{dx}(2x^2+5) + (2x^2+5)\frac{d}{dx}(3x+1)$$

$$= (3x+1)(4x) + (2x^2+5)(3)$$

$$f'(0) = \cancel{(3(0)+1)(4(0))} + (2(0)+5)(3)$$

$$= 15$$

\#

2.4.2 Derivative of a Quotient

Theorem 2.7 (The Quotient Rule) If f and g are both differentiable at x and if $g(x) \neq 0$, then f/g is differentiable at x and

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$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

ล่าง ล²

Example 17 Find dy/dx for y.

1. $y = \dfrac{x^2 - 1}{x^4 + 1}$ f(x) g(x)

Solution.

$$y' = \frac{(x^4+1)\frac{d}{dx}(x^2-1) - (x^2-1)\frac{d}{dx}(x^4+1)}{(x^4+1)^2}$$

$$= \frac{(x^4+1)(2x) - (x^2-1)(4x^3)}{(x^4+1)^2}$$

$$= \frac{2x^5 + 2x - 4x^5 + 4x^3}{(x^4+1)^2}$$

$$= \frac{-2x^5 + 4x^3 + 2x}{(x^4+1)^2}$$

$$= \frac{-2x(x^4 - 2x^2 - 1)}{(x^4+1)^2} \quad \#$$

2. $y = \dfrac{2x + 1}{\sqrt{x}}$

Solution.

$y = 2\sqrt{x} + \frac{1}{\sqrt{x}}$

$$y' = \frac{(\sqrt{x})\frac{d}{dx}(2x+1) - (2x+1)\frac{d}{dx}\sqrt{x}}{(\sqrt{x})^2}$$

$$= \frac{(\sqrt{x})(2) - (2x+1)\frac{1}{2\sqrt{x}}}{x}$$

$$= \frac{2\sqrt{x} - \frac{(2x+1)}{2\sqrt{x}}}{x} = \frac{4x - (2x+1)}{2x\sqrt{x}}$$

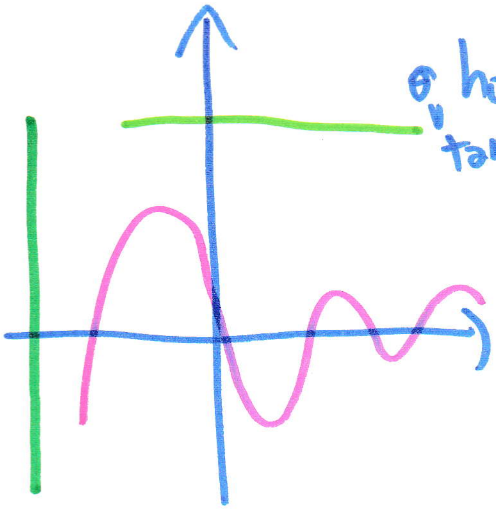
$$= \frac{2x-1}{2x^{\frac{3}{2}}} = \frac{1}{\sqrt{x}} - \frac{1}{2\sqrt{x^3}} \quad \#$$

(P.36)

แนวนอน

Example 13 At what points, if any, does the graph of $y = x^3 - 3x + 4$ have a horizontal tangent line?

vertical แนวตั้ง



∴ horizontal tangent , หาจุดที่มีความชัน = 0

↓

ดิฟแล้วเท่ากับ 0

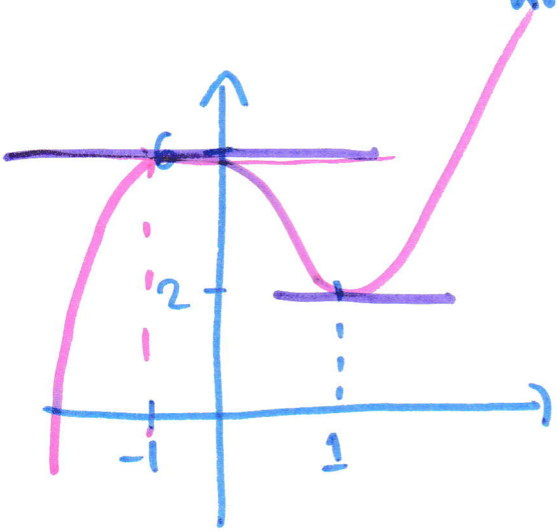
$\therefore y' = 3x^2 - 3 = 0$

พิจารณา $3x^2 - 3 = 0$

$x^2 - 1 = 0$

$(x-1)(x+1) = 0$

$x = 1, -1$



$\therefore$ ที่ $x = 1, -1$

มี horizontal tangent line.

2.3.1 Higher derivatives อนุพันธ์อันดับสูง

The derivative f' of a function f is itself a function and hence may have its own derivative. If f' is differentiable, then its derivative is denoted by f'' and is called the second derivative of f. As long as we have differentiability, we can continue the process of differentiating to obtain the third, the fourth, the fifth, and even higher derivatives of f. These successive derivatives are denoted by

the first derivative	y',	$f'(x)$,	$\frac{dy}{dx}$,	$\frac{d}{dx}[f(x)]$
the second derivative	y'',	$f''(x)$,	$\frac{d^2y}{dx^2}$,	$\frac{d^2}{dx^2}[f(x)]$
the third derivative	y''',	$f'''(x)$,	$\frac{d^3y}{dx^3}$,	$\frac{d^3}{dx^3}[f(x)]$
the fourth derivative	$y^{(4)}$,	$f^{(4)}(x)$,	$\frac{d^4y}{dx^4}$,	$\frac{d^4}{dx^4}[f(x)]$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A general nth order derivative	$y^{(n)}$,	$f^{(n)}(x)$,	$\frac{d^ny}{dx^n}$,	$\frac{d^n}{dx^n}[f(x)]$.

Example 14 If $f(x) = 6x^4 - 5x^3 + x^2 - 7x + 8$,then

$f'(x) = 24x^3 - 15x^2 + 2x - 7$

$f''(x) = 72x^2 - 30x + 2$

$f'''(x) = 144x - 30$

$f^{(4)}(x) = 144$

$f^{(5)}(x) = 0$

$f^{(500)}(x) = 0$

Ex 15 $\quad y = \dfrac{x+1}{x}$ หา $y^{(111)}$

$$y = 1 + \frac{1}{x} = 1 + x^{-1}$$

$$y' = 0 + (1)(-1)x^{-2} = -x^{-2}$$

$$y'' = (1)(-1)(-2)x^{-3} = (1)(2)x^{-3}$$

$$y''' = (1)(-1)(-2)(-3)x^{-4} = -(1)(2)(3)x^{-4}$$

$$y^{(4)} = -(1)(2)(3)(-4)x^{-5} = (1)(2)(3)(4)x^{-5}$$

$$\vdots$$

$$y^{(111)} = -\underbrace{(1)(2)\cdots(111)}x^{-112} = -111!\,x^{-112}$$

$$n! = 1\cdot 2\cdot 3\cdots(n-1)(n)$$

$$\boxed{y^{(n)} = (-1)^n n!\, x^{-(n+1)} = \frac{(-1)^n n!}{x^{n+1}}}$$

$$y^{(1000,000)} = \frac{(-1)^{1,000,000}(1,000,000)!}{x^{1000,000+1}}$$