3.2 Undefined matrix multiplication

$$\underbrace{\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\\10&11&12\end{pmatrix}}_{4\times 3}\underbrace{\begin{pmatrix}1&2&3&4\\5&6&7&8\end{pmatrix}}_{2\times 4}$$ is not defined

สลับที่

3.3 Is matrix multiplication commutative? .... NO.

Consider $A=\begin{bmatrix}3&1&-1\\2&0&3\end{bmatrix}$, $B=\begin{bmatrix}1&6\\3&-5\\-2&4\end{bmatrix}$ CHECK $A\times B = B\times A$

$$A\times B = \begin{bmatrix}3&1&-1\\2&0&3\end{bmatrix}_{2\times 3}\begin{bmatrix}1&6\\3&-5\\-2&4\end{bmatrix}_{3\times 2} = \begin{bmatrix}\ \end{bmatrix}_{2\times 2}$$

$$B\times A = \begin{bmatrix}1&6\\3&-5\\-2&4\end{bmatrix}_{3\times 2}\begin{bmatrix}3&1&-1\\2&0&3\end{bmatrix}_{2\times 3} = \begin{bmatrix}\ \end{bmatrix}_{3\times 3}$$

Note: We denote A^n the product of multiplying the matrix A to itself n times. For example, $A^2 = A\times A$ and $A^3 = A\times A\times A$.

$A=\begin{bmatrix}1&2\\2&1\end{bmatrix}$, $A^2=\begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}1&2\\2&1\end{bmatrix}$

4. Identity Matrices

$$\underbrace{\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}}_{A}\times\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{I}=\underbrace{\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}}_{A}=\underbrace{\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}}_{I}\times\underbrace{\begin{bmatrix}a_{11}&a_{12}&a_{13}\\a_{21}&a_{22}&a_{23}\\a_{31}&a_{32}&a_{33}\end{bmatrix}}_{A} =$$

In general,

$$\boxed{AI = A = IA}$$

for all square matrices A.

$I_1 = [1]_{1\times 1}$

$I_2 = \begin{bmatrix}1&0\\0&1\end{bmatrix}_{2\times 2}$

$I_5 = \begin{bmatrix}1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{bmatrix}_{5\times 5}$

5. Matrix Inverses

Let A be a square matrix of order n, and I be the identity matrix of order n. If there exists a matrix A^{-1} (called "A inverse") such that $A^{-1}A = AA^{-1} = I$. Then A^{-1} is called the inverse of A.

เอกฐาน

If no such matrix exists, then A is said to be a singular matrix. If A has an inverse, A is called a nonsingular matrix.

5.1 Inverse of a 2×2 matrix

There is a simple procedure to find the inverse of a two by two matrix. This procedure **only** works for the 2×2 case.

$A \times A^{-1} = I = A^{-1} \times A$

ในจำนวนจริง ภายใต้การบวก

$2+3 = 3+2$ สลับที่ภายใต้การบวก

$2+(2+3) = (2+2)+3$ เปลี่ยนกลุ่มภายใต้การบวก

$2+\boxed{0} = \boxed{0}+2 = 2$

$18+\boxed{0} = \boxed{0}+18 = 18$

} 0 เรียกว่าเป็นเอกลักษณ์ภายใต้บวก

อินเวอร์สการบวก

$2+\boxed{(-2)} = \boxed{0}$ ← เอกลักษณ์การบวก

$\frac{1}{2}+\boxed{\left(-\frac{1}{2}\right)} = \boxed{0}$ ← เอกลักษณ์การบวก

อินเวอร์สการบวก

ภายใต้การคูณ

$2\cdot 3 = 3\cdot 2$ สลับที่ภายใต้การคูณ

$2\cdot(2\cdot 3) = (2\cdot 2)\cdot 3$ เปลี่ยนกลุ่มภายใต้การคูณ

$2\cdot\boxed{1} = \boxed{1}\cdot 2 = 2$

$\frac{1}{2}\cdot\boxed{1} = \boxed{1}\cdot\frac{1}{2} = \frac{1}{2}$

} 1 เป็นเอกลักษณ์ภายใต้การคูณ

อินเวอร์สภายใต้การคูณ

$2\cdot\boxed{\frac{1}{2}} = \boxed{1}$ ← เอกลักษณ์การคูณ

Example 17 Find the inverse of $A=\begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix}$

$$A=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad A^{-1}=\frac{1}{(a_{11}a_{22}-a_{12}a_{21})}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

$$A^{-1}=\frac{1}{(2)(0)-(1)(-3)}\begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix}=\frac{1}{3}\begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix}$$

$$\boxed{A^{-1}=\begin{bmatrix} 0 & 1 \\ -\frac{1}{3} & \frac{2}{3} \end{bmatrix}}$$

$A^{-1}\cdot A = I = A\cdot A^{-1}$ ถ้ากลับไปคูณด้วยนะ:100 ~

5.2 Inverse of a general square matrix

1. Augment the matrix with the $n \times n$ identity matrix.
2. Use elementary row operations to transform the matrix on the left side of the vertical line to the $n \times n$ identity matrix. The row operations are used for the entire row, so that the matrix on the right hand side of the vertical line will be changed as well.
3. When the matrix on the left is transformed to the $n \times n$ identity matrix, the matrix on the right of the vertical line is the inverse.

Note: If, at some point, the matrix on the left satisfies properties 2 and 3 of reduced row echelon form and it is not an identity matrix, the initial matrix does not have an inverse.

Example 18 Find the inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{bmatrix}$.

ซ้าย ขวา

$$\underbrace{\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 2 & 5 & 3 & 0 & 1 & 0 \\ 1 & 0 & 8 & 0 & 0 & 1 \end{array}\right]}_{[A|I]} \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & -2 & 5 & -1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -2R_1+R_2 \to R_2 \\ -R_1+R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 0 & 1 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1 & -5 & 2 & 1 \end{array}\right] \begin{array}{l} R_3+R_1 \to R_1 \\ \\ 2R_2+R_3 \to R_3 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 8 & 0 & 0 & 1 \\ 0 & 1 & -3 & -2 & 1 & 0 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] (-1)R_3 \to R_3$$

$0-8(5)$ $0-8(-2)$ $1-8(-1)$

$-2+3(5)$ $1+3(-2)$ $0+3(-1)$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -40 & 16 & 9 \\ 0 & 1 & 0 & 13 & -5 & -3 \\ 0 & 0 & 1 & 5 & -2 & -1 \end{array}\right] \begin{array}{l} R_1-8R_3 \to R_1 \\ R_2+3R_3 \to R_2 \\ \\ \end{array}$$

$[I\,|\,A^{-1}]$

$$\therefore \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 8 \\ 1 & 0 & 8 \end{bmatrix}^{-1}=\begin{bmatrix} -40 & 16 & 9 \\ 13 & -5 & -3 \\ 5 & -2 & -1 \end{bmatrix}$$

$[A \mid I]$ $\qquad$ เป้าหมาย $[I \mid A^{-1}]$ $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & \cdot & \cdot & \cdot \end{array}\right]$

<u>Example 19</u> Find the inverse of $A = \begin{bmatrix} 1 & 6 & 4 \\ 2 & 4 & -1 \\ -1 & 2 & 5 \end{bmatrix}$.

$$\left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 2 & 4 & -1 & 0 & 1 & 0 \\ -1 & 2 & 5 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & -8 & -9 & -2 & 1 & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 2R_1 \to R_2 \\ R_3 + R_1 \to R_3 \end{array}$$

$4-2(6)$ $\quad -1-2(4)$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 6 & 4 & 1 & 0 & 0 \\ 0 & 1 & \frac{9}{8} & \frac{1}{4} & -\frac{1}{8} & 0 \\ 0 & 8 & 9 & 1 & 0 & 1 \end{array}\right] -\tfrac{1}{8}R_2 \to R_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & -\frac{11}{4} & -\frac{1}{2} & \frac{3}{4} & 0 \\ 0 & 1 & \frac{9}{8} & \frac{1}{4} & -\frac{1}{8} & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array}\right] \begin{array}{l} R_1 - 6R_2 \to R_1 \\ \\ R_3 - 8R_2 \to R_3 \end{array}$$

เกิด ภาว หายนะ

$\therefore$ A ไม่มีอินเวอร์สการคูณ

6. Basic Properties of Matrices

6.1 Addition Properties

เปลี่ยนกลุ่ม สลับที่

Associative: $(A + B) + C = A + (B + C)$

Commutative: $A + B = B + A$

Additive Identity: $A + 0 = 0 + A = A$

Additive Inverse: $A + (-A) = (-A) + A = 0$

เอกลักษณ์การบวกของเมทริกซ์ A คือ $[0]_{n \times n}$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\underline{0}} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

6.2 Multiplication Properties

Associative Property: $A(BC) = (AB)C$

Multiplicative identity: $AI = IA = A$

Multiplicative inverse: If A is a square matrix and A^{-1} exists, then $AA^{-1} = A^{-1}A = I$

6.3 Combined Properties

Left distributive: $A(B + C) = AB + AC$

Right distributive: $(B + C)A = BA + CA$

6.4 Equality

Addition: If A = B, then A + C = B + C

Left multiplication: If A = B, then CA = CB

Right multiplication: If A = B, then AC = BC

$A + C = B + C \Rightarrow A = B$ ✓

$\left.\begin{array}{l} CA = CB \\ AC = BC \end{array}\right\} \not\Rightarrow A = B$

สรุปไม่ได้นะ

7. Matrix Equations

Many of the basic properties of matrices are similar to the properties of real numbers.

Given an $n \times n$ matrix A and an $n \times p$ matrix B and a matrix X

$$\underline{A}X = B$$ (อยากได้ว่า $\boxed{X = \heartsuit}$)

$$\underbrace{A^{-1}(A}_{I}\, X) = A^{-1}B$$

$$(A^{-1}A)X = A^{-1}B$$

$$IX = A^{-1}B$$

$$\boxed{X = A^{-1}B}$$

Consider

$A\underline{X}=B$

coefficient matrix (matrix ส.ป.ส.)

variable matrix เมทริกซ์ตัวแปร

constant matrix

$$\underline{A}XB$$
$$A^{-1}(AX)=A^{-1}B$$
$$(A^{-1}A)X=A^{-1}B$$
$$(I_n)X=A^{-1}B$$
$$X=A^{-1}B$$

<u>Example 20</u> Use matrix inverses to solve the system

$$\begin{array}{rrrr} x & +y & +2z & =1 \\ 2x & +y & & =2 \\ x & +2y & +2z & =3 \end{array}$$

$$\begin{bmatrix} x+y+2z \\ 2x+y+0z \\ x+2y+2z \end{bmatrix}_{3\times 1} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$$

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix} \qquad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 0 \\ 1 & 2 & 2 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}_{B}$$

The inverse of A is $\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & \frac{-1}{4} & \frac{-1}{4} \end{bmatrix}$.

สมมติว่าไม่เชื่อ
ฝากหาอินเวอร์สของ A
เป็นการหอ /การบ้าน /การทดลองโด

$$\underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{X} = ♥$$

$$X = A^{-1} \cdot B$$

$$X = \begin{bmatrix} \left(\frac{1}{2}\right)(1) + \left(\frac{1}{2}\right)(2) - \frac{1}{2}(3) \\ (-1)(1) + (0)(2) + (1)(3) \\ \left(\frac{3}{4}\right)(1) + \left(\frac{-1}{4}\right)(2) + \left(\frac{-1}{4}\right)(3) \end{bmatrix}$$

$$X = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & 1 \\ \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} \end{bmatrix}_{3\times 3} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$$

$$X = \begin{bmatrix} 0 \\ 2 \\ -\frac{1}{2} \end{bmatrix}$$

Application

Example 21 Production scheduling: Labor and material costs for manufacturing two guitar models are given in the table below: Suppose that in a given week \$1800 is used for labor and \$1200 used for materials. How many of each model should be produced to use exactly each of these allocations?

Guitar model	Labor cost	Material cost
X	\$30	\$20
Y	\$40	\$30

Let x be the number of produced model X guit.

y " —————— " Y

$$30x + 40y = 1800 \Rightarrow 3x + 4y = 180$$
$$20x + 30y = 1200 \Rightarrow 2x + 3y = 120$$

ทด

$$\underbrace{\begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{X} = \underbrace{\begin{bmatrix} 180 \\ 120 \end{bmatrix}}_{B}$$

$$X = A^{-1} \cdot B$$

$$A^{-1} = \frac{1}{9-8}\begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$$

$$X = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 180 \\ 120 \end{bmatrix} = \begin{bmatrix} (3)(180) - (4)(120) \\ (-2)(180) + (3)(120) \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = X = \begin{bmatrix} 60 \\ 0 \end{bmatrix}$$

∴ ผลิตกีตาร์ X : 60 ชิ้น

และ ไม่ต้องผลิตกีตาร์ y.