Linear Programming (LP)

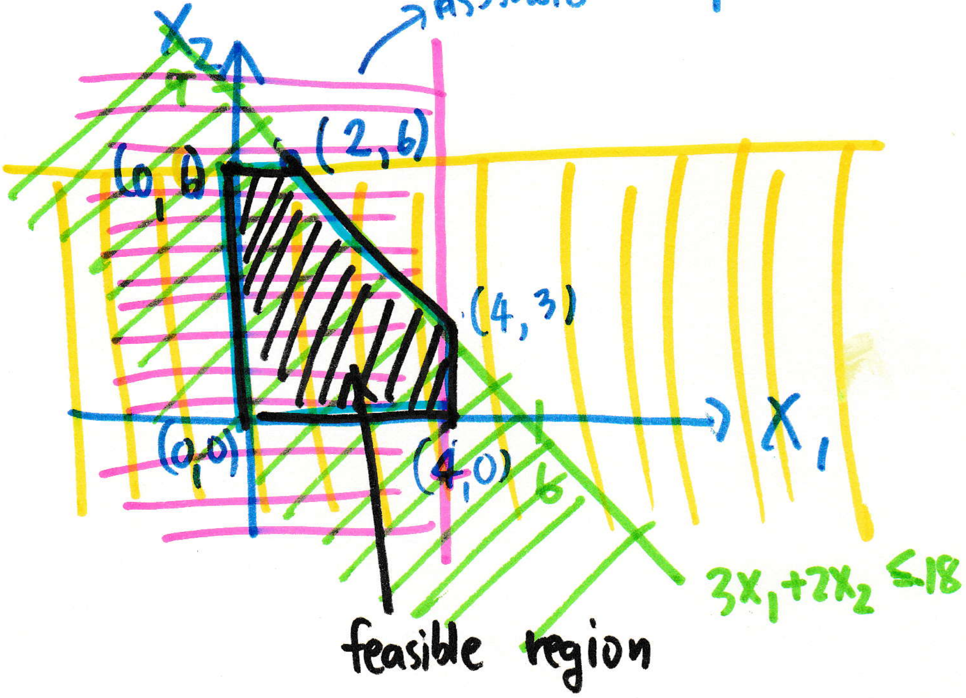
Objective function $\dfrac{\text{Max}}{\text{Min}}$ $Z = 3x_1 + 5x_2$

constraints:

$$\left.\begin{aligned} x_1 &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \\ x_1 \ge 0,\ x_2 &\ge 0 \end{aligned}\right\}$$

Solve : Geometric Method

ครึ่งระนาบ (halfplane)



x_1	x_2	$Z = 3x_1 + 5x_2$
0	0	0
0	6	30
2	6	36
4	3	27
4	0	12

ที่ (2,6) ให้ค่ามากสุดคือ 36

George B. Dantzig (1947)

Chapter 2: Linear Programming (LP) (part 2)

(some contents are taken from "Barnett/Ziegler/Byleen College Mathematics 12e" with modifications)

2.4 Simplex Method

The linear program is in the *standard form* if it seeks to maximize the objective linear function subject to a finite set of linear constraints in the form

$$a_1x_1 + a_2x_2 + \dots a_nx_n \le b, \text{ where } x_1, x_2 \dots x_n, b \ge 0.$$

The Simplex Method standard form

To solve an LP problem in the standard form by simplex method, we will use the following example to illustrate each step of the process.

$$\begin{array}{lrcrl}
\text{Max } Z = & 3x_1 & + & 5x_2 & \\
\text{s. t.} & x_1 & & & \le 4 \\
 & & & 2x_2 & \le 12 \\
 & 3x_1 & + & 2x_2 & \le 18 \\
 & \multicolumn{4}{l}{x_1 \ge 0,\ x_2 \ge 0}
\end{array}$$

This problem seeks for the maximum value and its 3 linear constraints are in the form $a_1x_1 + a_2x_2 + \dots a_nx_n \le b$, where $x_1, x_2 \dots x_n, b \ge 0$. Thus, it is in the standard form.

Step 1 Write the corresponding system of *constraint equations*, that is

1.1 All variables in the objective function are on the left-hand side of the equation.

1.2 All constraints must be equalities. For an inequality constraint, add *slack variables* to change it to an equality.

$$\begin{array}{lr}
Z - 3x_1 - 5x_2 & = 0 \\
x_1 + s_1 & = 4 \\
2x_2 + s_2 & = 12 \\
3x_1 + 2x_2 + s_3 & = 18
\end{array}$$

4 สมการ

6 ตัวแปร

slack variable : ตัวแปรช่วย

Step 2 Write the *simplex tableau* from the coefficient of the constraint equations with the objective function in the last row.

On the leftmost column, write the variable corresponding to each row (representing each equation), which is the slack variable in each equation or the optimal value variable.

ตัวแปรนอกฐาน non-basic variable

ตัวแปรอยู่ในฐาน basic variable

$x_1 = 0, x_2 = 0$, $s_1 = 4, s_2 = 12, s_3 = 18$

ตัวแปรอยู่ในฐาน

	x_1	x_2	s_1	s_2	s_3	
s_1	1	0	1	0	0	4
s_2	0	2	0	1	0	12
s_3	3	2	0	0	1	18
Z	−3	−5	0	0	0	0

Step 3 Find *pivot point* by

3.1 Choose the *entering column*, column containing a minimum negative coefficient in the last row (objective function).

3.2 For each row, compute the ratio between the last element (right-hand side of each constraint) and the coefficient in the entering column (only positive coefficients). Choose the *departing row*, a row with a minimum ratio.

3.3 The point on the entering column and the departing row is the *pivot* point. Replace the variable corresponding to the departing row by the variable corresponding to the entering column.

entering column

	x_1	x_2	S_1	S_2	S_3		
S_1	1	0	1	0	0	4	
S_2	0	2	0	1	0	12	$\frac{12}{2} = 6$
S_3	3	2	0	0	1	18	$\frac{18}{2} = 9$
Z	-3	-5	0	0	0	0	

departing row

pivot point

$x_1 = 0$

$x_2 = 0$

$z = 0$

Step 4 The following process is called *pivoting*. Unlike Gauss-Jordan elimination method, **pivoting _must_ be performed under the following rules only, not arbitrary row operations.**

4.1 Multiply the departing row by a proper k to change the pivot point to 1.

4.2 Use multiple of the departing row to change the other elements in the entering column to 0.

entering column

	x_1	x_2	S_1	S_2	S_3		
S_1	1	0	1	0	0	4	$\frac{4}{1} = 4$
x_2	0	1	0	$1/2$	0	6	
S_3	$3-2(0)$ 3	$2-2(1)$ 0	$0-2(0)$ 0	$0-2(\frac{1}{2})$ -1	$1-2(0)$ 1	$18-2(6)$ 6	$\frac{6}{3} = 2$
Z	$-3+5(0)$ -3	$-5+5(1)$ 0	$0+5(0)$ 0	$0+5(\frac{1}{2})$ $5/2$	$0+5(0)$ 0	$0+5(6)$ 30	

pivot

$x_1 = 0$

$x_2 = 6$

$z = 30$

$\frac{1}{2}R_2 \to R_2$

$R_3 - 2R_2 \to R_3$

$R_4 + 5R_2 \to R_4$

Step 5 Repeat 3 and 4 until all elements in the last rows become nonnegative. The rightmost column gives the value of each variable in the optimal solution.

If a variable does not appear in the leftmost column at the end of the process, that variable is 0 in the optimal solution.

Note: 1) Pivot points are always positive and not in the last rows.

2) If we cannot find any pivot points at some step before all elements in the last rows becoming nonnegative, there are no optimal solutions.

	x_1	x_2	S_1	S_2	S_3	
S_1	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
x_2	0	1	0	$1/2$	0	6
x_1	1	0	0	$-\frac{1}{3}$	$\frac{1}{3}$	2
Z	$-3+3(1)$ 0	$0+3(0)$ 0	$0+3(0)$ 0	$5/2+3(-\frac{1}{3})$ $3/2$	$0+3(\frac{1}{3})$ 1	$30+3(2)$ 36

$R_1 - R_3 \to R_1$

$\frac{1}{3}R_3 \to R_3$

$R_4 + 3R_3 \to R_4$

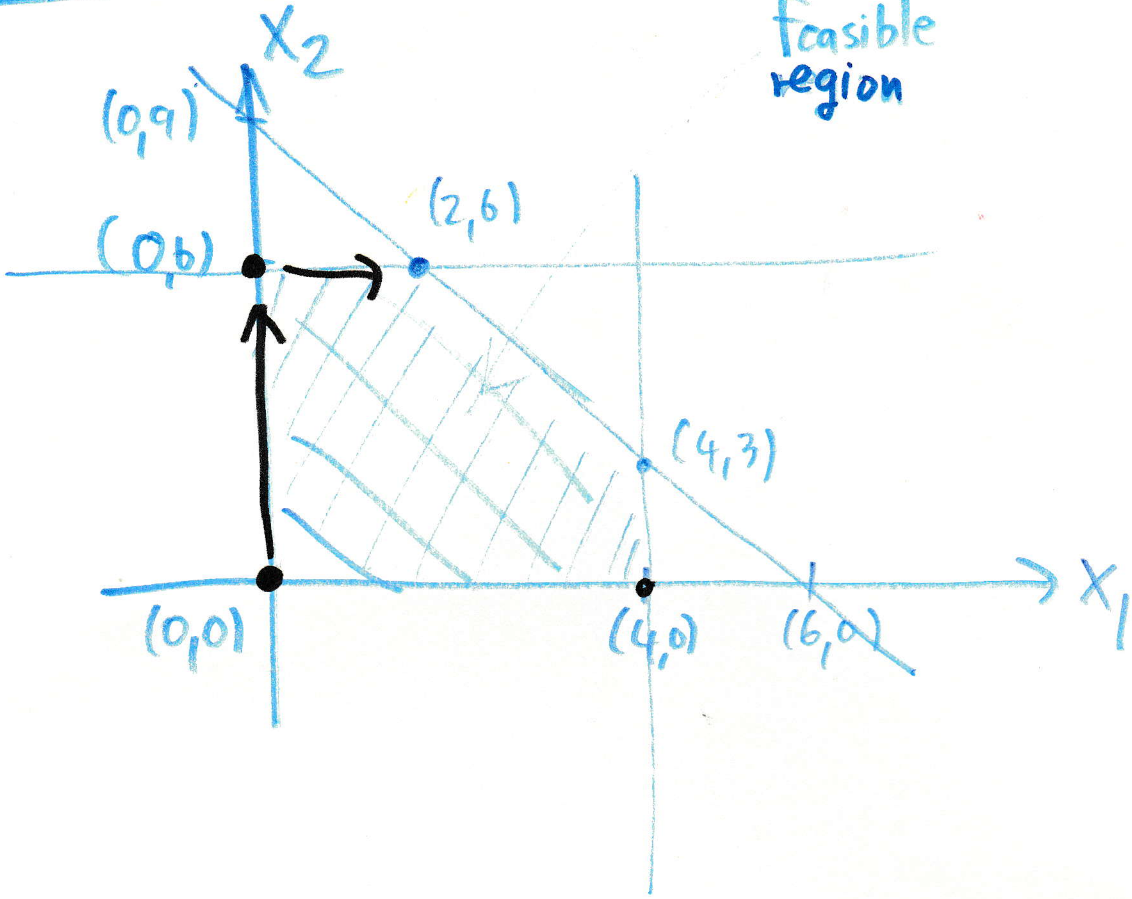
Optimal solution is $(x_1, x_2) = (2, 6)$ Optimal value is 36

$x_1 = 2$

$x_2 = 6$

$z = 36$

Intepretation



Example 2.4.1. **(Multiple Optimal Solutions)**
Use the simplex method to find the maximum value of the following optimization problem

Max $\quad Z = 4x_1 + 14x_2$
Subject to $\quad 2x_1 + 7x_2 \leq 21$
$\quad 7x_1 + 2x_2 \leq 21$
$\quad x_1 \geq 0,\ x_2 \geq 0$

$2x_1 + 7x_2 + s_1 = 21$
$7x_1 + 2x_2 + s_2 = 21$
$z - 4x_1 - 14x_2 = 0$

entering column

departing row

$x_1 = 0$
$x_2 = 0$

	x_1	x_2	s_1	s_2	
s_1	2	7 pivot	1	0	21 $= \frac{21}{7} = 3$
s_2	7	2	0	1	21 $\frac{21}{2} = 10.5$
z	-4	-14	0	0	0
$\cancel{s_1}$ x_2	$\frac{2}{7}$	1	$\frac{1}{7}$	0	3
s_2	$7-2(\frac{2}{7})$ $45/7$	$2-2(1)$ 0	$0-2(\frac{1}{7})$ $-2/7$	$1-2(0)$ 1	$21-2(3)$ 15
z	$-4+14(\frac{2}{7})$ 0	$-14+14(1)$ 0	$0+14(\frac{1}{7})$ 2	$0+14(0)$ 0	$0+14(3)$ 42

$\frac{1}{7}R_1 \to R_1$
$R_2 - 2R_1 \to R_2$
$R_3 + 14R_1 \to R_3$

$x_1 = 0$
$x_2 = 3$

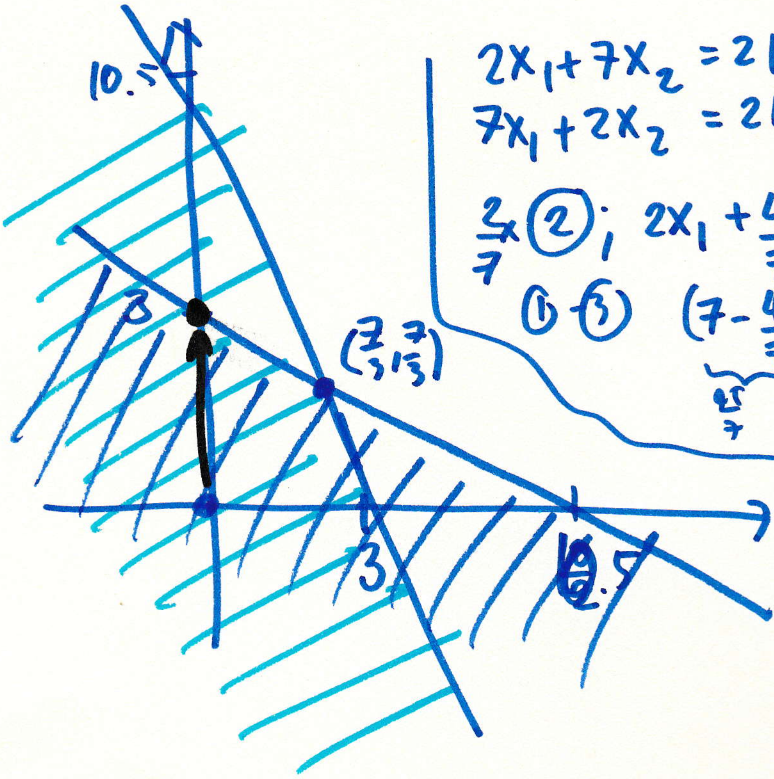
Optimal solution is $(x_1, x_2) = (0, 3)$ Optimal value is 42

Example 2.4.2. (Unboundedness)
Use the simplex method to find the maximum value of the following optimization problem

[illegible]
[illegible]
[illegible]
[illegible]

Max $\quad Z = 2x_1 + x_2$
Subject to $\quad x_1 - x_2 \leq 10$
$\quad 2x_1 - x_2 \leq 40$
$\quad x_1 \geq 0,\ x_2 \geq 0$

Optimal solution is________________________ Optimal value is __________



$2x_1+7x_2=21$ — ①
$7x_1+2x_2=21$ — ②

$\frac{2}{7}\times$ ②; $2x_1+\frac{4}{7}x_2=\frac{42}{7}$ — ③

① − ③ $\;(7-\frac{4}{7})x_2=15,$

$\underbrace{(7-\frac{4}{7})}_{\frac{45}{7}}x_2=15$

$x_2=\frac{15}{45}\times 7=\frac{7}{3}$

$2x_1+7(\frac{7}{3})=21$

$2x_1=21-\frac{49}{3}$

$2x_1=\frac{63-49}{3}$

$x_1=\frac{14}{2\cdot 3}=\frac{7}{3}.$

(x,y)	$z=4x_1+14x_2$
(0,0)	0
(0,3)	42
(3,0)	12
$(\frac{7}{3},\frac{7}{3})$	42