ฤบทอนุ 1

(1) $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$

Special

$\frac{d}{dx}(\sqrt{u}) = \frac{1}{2\sqrt{u}}\frac{du}{dx}$

(2) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(3.1) $\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

(3.2) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

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(4) $\frac{d}{dx}(\sin u) = \cos u\frac{du}{dx}$

$\frac{d}{dx}(\cot u) = -\operatorname{cosec}^2 u\frac{du}{dx}$

$\frac{d}{dx}(\cos u) = -\sin u\frac{du}{dx}$

$\frac{d}{dx}(\sec u) = \sec u \tan u\frac{du}{dx}$

$\frac{d}{dx}(\tan u) = \sec^2 u\frac{du}{dx}$

$\frac{d}{dx}(\operatorname{cosec} u) = -\operatorname{cosec} u \cot u\frac{du}{dx}$

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(5) $\frac{d}{dx}(a^u) = a^u \ln a\frac{du}{dx}$

$\frac{d}{dx}(\log_a u) = \frac{1}{u\ln a}\frac{du}{dx}$

$\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$

$\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$

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(6) $\frac{d}{dx}(\sin^{-1}u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

$\frac{d}{dx}(\sec^{-1}u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

$\frac{d}{dx}(\cos^{-1}u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

$\frac{d}{dx}(\operatorname{cosec}^{-1}u) = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

$\frac{d}{dx}(\tan^{-1}u) = \frac{1}{1+u^2}\frac{du}{dx}$

$\frac{d}{dx}(\cot^{-1}u) = -\frac{1}{1+u^2}\frac{du}{dx}$

Example: Find the derivative of $y = x^x$ (หา $\frac{dy}{dx}$)

Take log ทั้งสองข้าง

$\ln(x^a) = a\ln x$

$$\ln y = \ln(x^x)$$

$$\ln y = x\ln x$$

$$\frac{d}{dx}(\ln y) = \frac{d}{dx}(x\ln x)$$

$$\frac{1}{y}\frac{dy}{dx} = x\frac{d}{dx}(\ln x) + \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = \cancel{x}\left(\frac{1}{\cancel{x}}\right) + \ln x$$

$$\frac{1}{y}\boxed{\frac{dy}{dx}} = 1 + \ln x$$

$$\frac{dy}{dx} = y(1+\ln x)$$

$$\frac{dy}{dx} = x^x(1+\ln x) \quad \#$$

Example 9 Find the derivative of $y = (x^2 + 1)^{\sin x}$.

Take $\underline{\ln}$ (log) ทั้งสองข้าง

$$\ln y = \ln (x^2+1)^{\sin x}$$

ห้ามทำ!

$$\cancel{\ln(x^2) + \ln(1)}$$

$$\ln y = (\sin x) \ln (x^2+1)$$

$$\frac{d}{dx}(\ln y) = \sin x \frac{d}{dx}(\ln(x^2+1)) + \ln(x^2+1)\frac{d}{dx}(\sin x)$$

$$\frac{1}{y}\frac{dy}{dx} = (\sin x)\frac{1}{x^2+1}\frac{d}{dx}(x^2+1) + \ln(x^2+1)(\cos x)$$

$$\frac{1}{y}\boxed{\frac{dy}{dx}} = \frac{\sin x}{x^2+1}(2x) + (\ln(x^2+1))(\cos x)$$

$$\frac{dy}{dx} = y\left[\frac{2x\sin x}{x^2+1} + (\cos x)(\ln(x^2+1))\right]$$

$$\boxed{\frac{dy}{dx} = (x^2+1)^{\sin x}\left[\frac{2x\sin x}{x^2+1} + (\cos x)(\ln(x^2+1))\right]}$$

#

3.4 Related rates (อัตราสัมพัทธ์)

In this section, one tries to find the rate at which some quantity is changing by relating itself to other quantities, whose rates of change are known.

Revision: Chain Rule

THEOREM 2.9 (THE CHAIN RULE) If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover, if

$$y = f(g(x)) \text{ and } u = g(x)$$

then $y = f(u)$ and

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.}$$

Example 13 Suppose that x, y are differentiable functions of t and are related by $y = x^4 + 5$. Find $\boxed{\frac{dy}{dt}|_{t=2}}$ if $x(2) = 1$ and $\frac{dx}{dt}|_{t=2} = -2$.

$$\boxed{y = x^4 + 5}$$

อยากได้ $\frac{dy}{dt}$

$$\frac{dy}{dt} = \frac{d}{dt}(x^4+5)$$

$$\frac{dy}{dt} = 4x^3\frac{dx}{dt}$$

$$\frac{dy}{dt} = 4(x(t))^3\frac{dx}{dt}$$

$$\frac{dy}{dt}\Big|_{t=2} = 4(x(2))^3\frac{dx}{dt}\Big|_{t=2}$$

$$= 4(1)^3(-2) = -8$$

มองปัญหาแบบกฎลูกโซ่

$$\boxed{\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}}$$

$y = x^4 + 5$

$\frac{dy}{dx} = 4x^3$

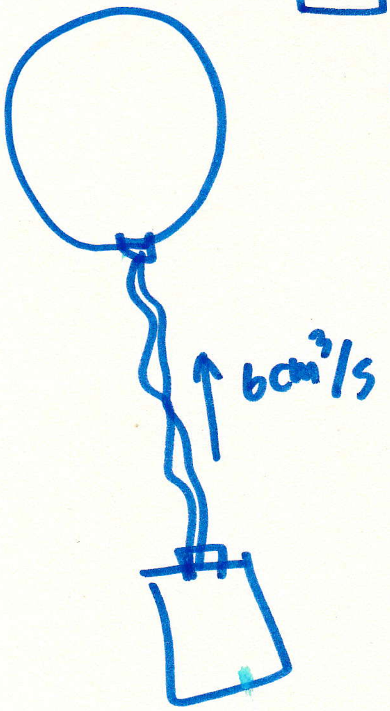
$$\left.\frac{dy}{dt}\right|_{t=2} = \left.\frac{dy}{dx}\right|_{t=2} \cdot \left.\frac{dx}{dt}\right|_{t=2}$$

$$= 4(x(2))^3(-2)$$

$$= 4(+1)^3(-2) = -8$$

Example 14 Air is being pumped into a spherical balloon at the rate of 6 cubic centimetres per second. What is the rate of change of the radius at the instant the volume equals 36π? The volume of a sphere of radius r is $\boxed{\frac{4\pi r^3}{3}}$.

$\frac{dr}{dt}$



สมมติว่า r เป็นรัศมีของทรงกลม

V เป็นปริมาตรของทรงกลม

$$\therefore \boxed{V = \frac{4}{3}\pi r^3}$$

$$\frac{dV}{dt} = \frac{4\pi}{3}\frac{d}{dt}(r^3)$$

$$\frac{dV}{dt} = \frac{4\pi}{\cancel{3}}(\cancel{3}r^2)\boxed{\frac{dr}{dt}}$$

$$\frac{dV}{dt} = 4\pi r^2 \boxed{\frac{dr}{dt}}$$

$$\boxed{\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}} \to 6\ cm^3/s$$

$$\boxed{V = 36\pi}$$

$$V = \frac{4}{3}\pi r^3$$

$$\overset{9}{\cancel{36}}\cancel{\pi} = \frac{\cancel{4}}{3}\cancel{\pi} r^3$$

$$\boxed{r = 3}$$

$$\left.\frac{dr}{dt}\right|_{r=3} = \frac{1}{\underset{2}{\cancel{4}}\pi(3)^{\cancel{2}}}(\overset{3}{\cancel{6}})\ cm/s$$

$$\boxed{\left.\frac{dr}{dt}\right|_{r=3} = \frac{1}{6\pi}\ cm/s}$$

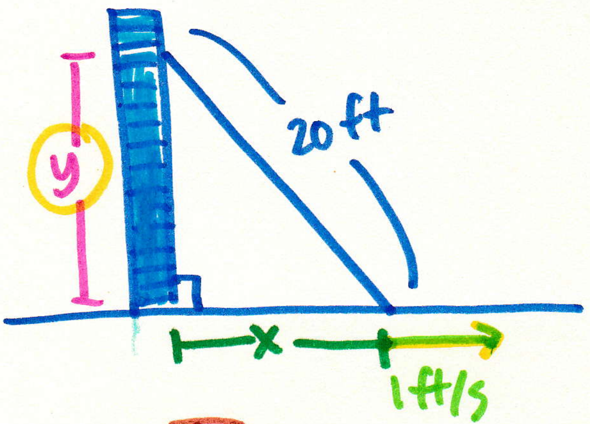
$\therefore$ เมื่อบอลลูนพองโตมีปริมาตร 36π

รัศมีเพิ่มขึ้นด้วยอัตรา $\frac{1}{6\pi}$ cm/s. #

THE STEPS IN SOLVING A RELATED RATES PROBLEM:

- Decide what the two variables are.
- Find an equation relating them.
- Take $\frac{d}{dt}$ of both sides.
- Plug in all known values at the instant in question.
- Solve for the unknown rate.

Example 15 A ladder 20 feet long is placed against a wall. The foot of the ladder begins to slide away from the wall at the rate of 1 ft/sec. How fast is the top of the ladder sliding down the wall when the foot of the ladder is 12 feet from the wall?



สมมติว่า x คือระยะที่วัดจากตีนบันไดไปยังกำแพง
y คือความสูงจากพื้นไปยังปลายบันได (แนวกำแพง)

ความสัมพันธ์ระหว่าง x และ y พีทาโกรัส

$$\boxed{x^2+y^2=(20)^2}$$

$$\frac{d(x^2)}{dt}+\frac{d(y^2)}{dt}=\frac{d}{dt}(20)^2$$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

$$\frac{dy}{dt}=-\frac{\cancel{2}x}{\cancel{2}y}\frac{dx}{dt}$$

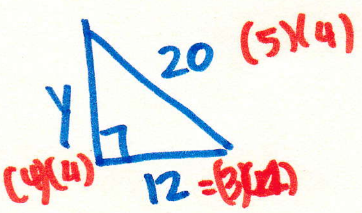
$$\boxed{\frac{dy}{dt}=-\frac{x}{y}\frac{dx}{dt}}$$ 1 ft/s

เป้าหมาย $\boxed{\frac{dy}{dt}}$

รู้ว่า $x=12$ หา y ได้จาก

$$(12)^2+y^2=(20)^2$$

$$y=16$$



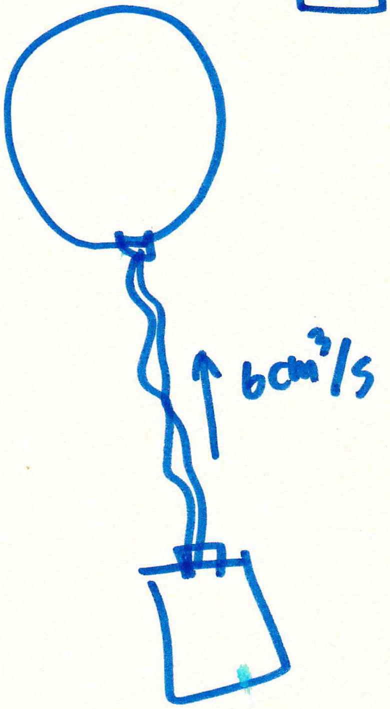
$$\left.\frac{dy}{dt}\right|_{\substack{x=12\\y=16}}=\left(-\frac{12}{16}\right)(1)\text{ ft/s}$$

$$=-\frac{3}{4}\text{ ft/s.}$$

∴ ความสูงจากพื้นถึงยอดบันไดลดลงด้วยอัตรา 3/4 ft/s.

Example 14 Air is being pumped into a spherical balloon at the rate of 6 cubic centimetres per second. What is the rate of change of the radius at the instant the volume equals 36π? The volume of a sphere of radius r is $\boxed{\dfrac{4\pi r^3}{3}}$.

$\frac{dr}{dt}$



สมมติว่า r เป็นรัศมีของทรงกลม

V เป็นปริมาตรของทรงกลม

$$\therefore \boxed{V = \frac{4}{3}\pi r^3}$$

$$\frac{dV}{dt} = \frac{4\pi}{3}\frac{d}{dt}(r^3)$$

$$\frac{dV}{dt} = \frac{4\pi}{\cancel{3}}(\cancel{3}r^2)\boxed{\frac{dr}{dt}}$$

$$\frac{dV}{dt} = 4\pi r^2\boxed{\frac{dr}{dt}}$$

$\rightarrow$ 6 cm³/s

$$\boxed{\frac{dr}{dt} = \frac{1}{4\pi r^2}\frac{dV}{dt}}$$

$$\boxed{V = 36\pi}$$

$$V = \frac{4}{3}\pi r^3$$

$$\overset{9}{\cancel{36}}\cancel{\pi} = \frac{\cancel{4}}{3}\cancel{\pi} r^3$$

$$\boxed{r = 3}$$

$$\left.\frac{dr}{dt}\right|_{r=3} = \frac{1}{\underset{2}{\cancel{4}}\pi(3)^2}(\overset{3}{\cancel{6}}) \text{ cm/s}$$

$$\boxed{\left.\frac{dr}{dt}\right|_{r=3} = \frac{1}{6\pi} \text{ cm/s}}$$

$\therefore$ เมื่อบอลลูนพองโตมีปริมาตร 36π

รัศมีเพิ่มขึ้นด้วยอัตรา $\frac{1}{6\pi}$ cm/s. #

(Additional Problem)

A particle is moving along the curve $\boxed{\frac{x^2}{2} + \frac{y^2}{8} = 1}$, where x and y are differentiable functions of time t. Find the rate of change of y with respect to time at the point $(1, 2)$ when x is decreasing at a rate of 1 unit / second.

อนุภาคหนึ่งกำลังเคลื่อนที่ตามเส้นโค้ง $\frac{x^2}{2} + \frac{y^2}{8} = 1$ โดยที่ x และ y เป็นฟังก์ชันของเวลา t ที่หาอนุพันธ์ได้ จงหาอัตราการเปลี่ยนแปลงของ y เทียบกับเวลา ที่จุด (1, 2) เมื่อ x กำลังลดลงด้วยอัตรา 1 หน่วยต่อวินาที