<u>ทบทวน</u> $\longrightarrow$ Geometric Method

$\quad\;\;\, \hookrightarrow$ Simplex Method

<u>Standard Problem</u>

$$\max Z = c_1x_1 + \dots + c_nx_n$$

s.t. (subject to)

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\leq b_1 \\
a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\leq b_2 \\
\vdots \qquad\qquad\qquad & \quad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\leq b_m \\
x_1 \geq 0,\ x_2 \geq 0,\ \dots,\ x_n &\geq 0
\end{aligned}$$

2.5 Dual Problem: Minimization With Problem Constraints of the Form ≥

Formation of the Dual Problem

0. Write an LP problem in the form of minimization problem with ≥ problem constraints.
1. Use the coefficients and constants in the problem constraints and the objective function to form a matrix *A* with the coefficients of the objective function in the last row.
2. Interchange the rows and columns of matrix *A* to form the matrix A^T, the transpose of *A*.
3. Use the rows of A^T to form a maximization problem with ≤ problem constraints.

Given a maximization problem with ≤ problem constraints, we can do the similar process to get the dual problem in the minimization form.

Example 2.5.1 Find a dual problem of the following

$$
\begin{aligned}
\text{Min } C &= 2x_1 + 3x_2 + x_3 \\
\text{s. t. } & x_1 + 4x_2 + 2x_3 \ge 8 \\
& x_1 - 2x_2 + x_3 \le 6 \qquad -x_1 + 2x_2 - x_3 \ge -6 \\
& x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0
\end{aligned}
$$

$$
A = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 4 & 2 & 8 \\ -1 & 2 & -1 & -6 \\ \hline 2 & 3 & 1 & \end{array}\right] \end{array}
\qquad
A^T = \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 4 & 2 & 3 \\ 2 & -1 & 1 \\ \hline 8 & -6 & \end{array}\right] \end{array}
$$

$$
\begin{aligned}
\text{Max } C' &= 8y_1 - 6y_2 \\
& y_1 - y_2 \le 2 \\
& 4y_1 + 2y_2 \le 3 \\
& 2y_1 - y_2 \le 1 \\
& y_1 \ge 0,\ y_2 \ge 0
\end{aligned}
$$

Example 2.5.2 Find the Optimal solution of the following optimization problem

$$
\begin{aligned}
\text{Min } C &= 14x_1 + 9x_2 + 21x_3 \\
\text{s. t. } & x_1 + x_2 + x_3 \ge 4 \\
& 2x_1 + x_2 + x_3 \ge 16 \\
& x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0
\end{aligned}
$$

$$
A = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 4 \\ 2 & 1 & 1 & 16 \\ \hline 14 & 9 & 21 & \end{array}\right] \end{array}
\qquad
A^T = \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & 2 & 14 \\ 1 & 1 & 9 \\ 1 & 1 & 21 \\ \hline 4 & 16 & \end{array}\right] \end{array}
$$

$$
\left.
\begin{aligned}
\text{Max } C' &= 4y_1 + 16y_2 \\
\text{s.t. } & y_1 + 2y_2 \le 14 \\
& y_1 + y_2 \le 9 \\
& y_1 + y_2 \le 21 \\
& y_1 \ge 0,\ y_2 \ge 0
\end{aligned}
\right\}
\qquad
\begin{aligned}
y_1 + 2y_2 + s_1 &= 14 \\
y_1 + y_2 + s_2 &= 9 \\
y_1 + y_2 + s_3 &= 21 \\
-4y_1 - 16y_2 + C' &= 0
\end{aligned}
$$

$S_1=8,\ S_2=0,\ S_3=0$

$\Rightarrow x_1=8,\ x_2=0,\ x_3=0$

	y_1	y_2	S_1	S_2	S_3		
S_1	1	2	1	0	0	14	$=\frac{14}{2}$
S_2	1	1	0	1	0	9	$=\frac{9}{1}$
S_3	1	1	0	0	1	21	$=\frac{21}{1}$
C'	−4	−16	0	0	0	0	
$\frac{1}{2}R_1 \to R_1$ Y_2	1/2	1	1/2	0	0	7	$y_1=0$
$R_2-R_1 \to R_2$ S_2	$1-\frac{1}{2}$ → 1/2	$1-1$ → 0	$0-1/2$ → −1/2	$1-0$ → 1	$0-0$ → 0	$9-7$ → 2	$y_2=7$
$R_3-R_1 \to R_3$ S_3	$1-1/2$ → 1/2	$1-1$ → 0	$0-1/2$ → −1/2	$0-0$ → 0	$1-0$ → 1	$21-7$ → 14	
$R_4+16R_1 \to R_4$ C'	$-4+16(1/2)$ → 4	$-16+16(1)$ → 0	$0+16(1/2)$ → 8	$0+16(0)$ → 0	$0+16(0)$ → 0	$0+16(7)$ → 112	

The dual problem
Optimal solution is $(y_1,y_2)=(0,7)$ Optimal value is 112

The original problem
Optimal solution is $(x_1,x_2,x_3)=(8,0,0)$ Optimal value is 112

2.6 Max and Min with Mixed Problem constraints

Maximization
Constraint equations of the modified problem
Step 1: If any problem constraints have negative constants on the right side, multiply both sides by -1 to obtain a constraint with a nonnegative constant.
Remember to reverse the direction of the inequality if the constraint is an inequality.
Step 2: Add a slack variable for each constraint of the form ≤.
Step 3: Add an *artificial variable* - a *surplus variable* and in each ≥ constraint.
Step 4: Add an artificial variable in each = constraint.
Step 5: For each artificial variable *a*, add the *penalty cost*, $-Ma$ to the objective function. Use the same constant *M* for all artificial variables. Move all variables to the left-hand side.

Example 2.6.1 Find the constraint equations of the modified problem of the following linear programming problem (Do not attempt to solve the problem.)

Max $P = 2x_1 + x_2$ $-Ma_1$
s. t. $x_1 + x_2 \le 10$
$\boxed{-x_1 + x_2 \ge 2}$
$x_1 \ge 0, x_2 \ge 0$

$-x_1+x_2 \geq 2$

กำหนดไว้ใหญ่มากๆ

slack variable

$x_1+x_2 \boxed{+S_1} \qquad =10$ Artificial variable

$-x_1+x_2 \boxed{-S_2} +a_1 = 2$ ~~+~~

Surplus variable

Max $P = 2x_1+x_2-Ma_1$

$x_1+x_2+S_1 \qquad =10$

$-x_1+x_2 \qquad -S_2+a_1=2$

$x_1, x_2, S_1, S_2, a_1 \geq 0$

Theorem (Fundamental Principle of Duality)

A minimization problem has a solution if and only if its dual problem has a solution.

If a solution exists, then the optimal value of the minimization prob. is the same as the optimal value of the dual problem.

HW การบ้าน / การหอ / การเลื่อนโต / การอันดับตามนัด

$$\text{Min } C = 16\cancel{0}x_1 + 45x_2$$

$$\text{s.t.} \quad 2x_1 + 5x_2 \geq 50$$

$$x_1 + 3x_2 \geq 27$$

$$x_1, x_2 \geq 0$$

โดย Simplex Method

แล้วเทียบกับ geometric method

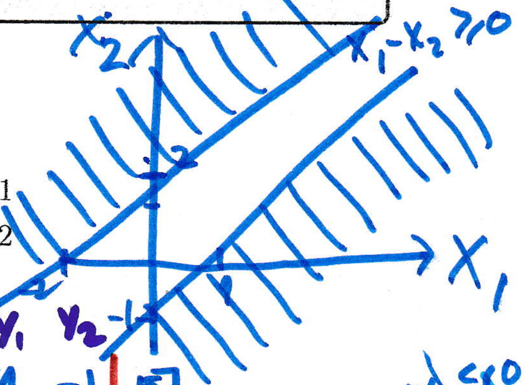
ทบทวนปัญหา Minimization โดย Simplex Method

ชื่อ-สกุล เฉลย. รหัสนักศึกษา.......................... ลำดับที่..........

กำหนดปัญหากำหนดการเชิงเส้น

$$\min C = 5x_1 + 10x_2$$

Subject to
$$x_1 - x_2 \geq 1$$
$$-x_1 + x_2 \geq 2$$
$$x_1, x_2 \geq 0$$



1. สร้างเมทริกซ์ A ที่สอดคล้องกับปัญหาค่าต่ำสุด แล้วคำนวณ A^T

$$A = \begin{array}{c} \begin{array}{cc} x_1 & x_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -1 & 1 \\ -1 & 1 & 2 \\ \hline 5 & 10 & \end{array}\right] \end{array} \qquad A^T = \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -1 & 5 \\ -1 & 1 & 10 \\ \hline 1 & 2 & 0 \end{array}\right] \end{array}$$

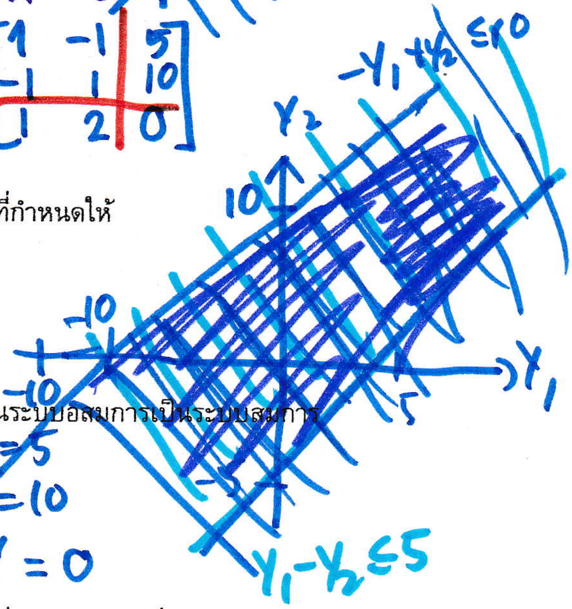
2. จากเมทริกซ์ A^T ให้เขียนปัญหาควบคู่ (Dual Problem) ของปัญหาที่กำหนดให้

Max $C' = y_1 + 2y_2$

s.t. $y_1 - y_2 \leq 5$

$-y_1 + y_2 \leq 10$

$y_1 \geq 0, y_2 \geq 0$



3. จากปัญหาควบคู่ ให้กำหนดตัวแปรช่วย (Slack Variable) เพื่อเปลี่ยนระบบอสมการเป็นระบบสมการ

$y_1 - y_2 + s_1 = 5$

$-y_1 + y_2 + s_2 = 10$

$-y_1 - 2y_2 + C' = 0$

4. สร้างตาราง Simplex แล้วแก้ปัญหาโดยใช้วิธี Simplex ตามขั้นตอนเพื่อหาค่าเหมาะที่สุด

		y_1	y_2	s_1	s_2	
	s_1	1	-1	1	0	5
	s_2	-1	1	0	1	10
	C'	-1	-2	0	0	0
$R_1 + R_2 \to R_1$	s_1	0	0	1	0	15
	y_2	-1	1	0	1	10
$R_3 + 2R_2 \to R_3$	C'	-3	0	0	2	20

5. สรุปคำตอบและตีความหมายคำตอบ (วาดกราฟเพื่อเทียบกับ Geometric Method)

∴ ในระบบ dual ไม่มี optimal solution

ดังนั้น ปัญหาหลัก (ดั้งเดิม) ไม่มี optimal solution.

Example 2.6.2 Find the constraint equations of the modified problem of the following linear programming problem (Do not attempt to solve the problem.).

$$
\begin{array}{lrl}
\text{Max} & P = & 2x_1 + 5x_2 + 3x_3 \quad -Ma_1 - Ma_2 - Ma_3 \\
\text{s. t.} & & x_1 + 2x_2 - x_3 \le 7 \\
& & -x_1 + x_2 - 2x_3 \le -5 \quad \Rightarrow \quad x_1 - x_2 + 2x_3 \geq 5 \\
& & x_1 + 4x_2 + 3x_3 \ge 1 \\
& & 2x_1 - x_2 - 4x_3 = 6 \\
& & x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0
\end{array}
$$

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + s_1 \quad &= 7 \\
x_1 - x_2 + 2x_3 \quad - s_2 + a_1 &= 5 \\
x_1 + 4x_2 + 3x_3 \quad - s_3 + a_2 &= 1 \\
2x_1 - x_2 - 4x_3 \quad + a_3 &= 6 \\
-2x_1 - 5x_2 - 3x_3 + Ma_1 + Ma_2 + Ma_3 + P &= 0
\end{aligned}
$$

$$x_1, x_2, x_3, s_1, s_2, s_3, a_1, a_2, a_3 \geq 0$$

2.7 Big M Method:

1. Form the preliminary simplex tableau for the modified problem.
2. Use row operations to eliminate the *M*s in the bottom row of the preliminary simplex tableau in the columns corresponding to the artificial variables. The resulting tableau is the initial simplex tableau.
3. Solve the modified problem by applying the simplex method to the initial simplex tableau found in the second step.
4. Relate the optimal solution of the modified problem to the original problem.
   A) if the modified problem has no optimal solution, the original problem has no optimal solution.
   B) if all artificial variables are 0 in the optimal solution to the modified problem, delete the artificial variables to find an optimal solution to the original problem
   C) if any artificial variables are nonzero in the optimal solution, the original problem has no optimal solution.

Example 2.7.1 Solve the following linear programming problem.

$$
\begin{array}{lrl}
\text{Max} & P = & 2x_1 + x_2 \\
\text{s. t.} & & x_1 + x_2 \le 10 \\
& & -x_1 + x_2 \ge 2 \\
& & x_1 \ge 0, x_2 \ge 0
\end{array}
$$

s.t. $y_1, y_2 \geq 0$.

Theorem: (Fundamental Principle of Duality)

A minimization problem has a solution if and only if its dual problem has a solution. If a solution exists, then the optimal value of the minimization problem is the same as the optimal value of the dual problem.

Ex.

Primal

Min $C = 16x_1 + 45x_2$

s.t. $2x_1 + 5x_2 \geq 50$

$x_1 + 3x_2 \geq 27$

$x_1, x_2 \geq 0$

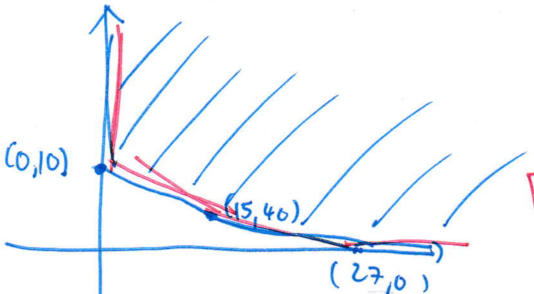
Dual Problem

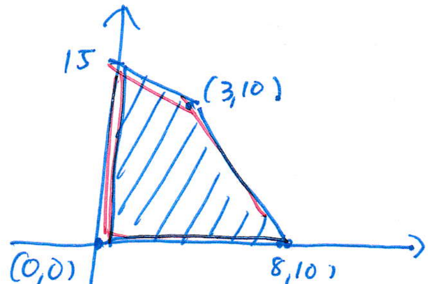
Max $P = 50y_1 + 27y_2$

s.t. $2y_1 + y_2 \leq 16$

$5y_1 + 3y_2 \leq 45$

$y_1, y_2 \geq 0$





(x_1, y_1)	$C = 16x_1 + 45x_2$	
(0,10)	450	
(15,4)	420	min
(27,0)	472	

However, the optimal does not garuntee that they are same!

(y_1, y_2)	$P = 50y_1 + 27y_2$	
(0,0)	0	
(0,15)	405	
(3,10)	420	max
(8,0)	400	