ทบทวน $\max Z = 3x_1 + 5x_2$

subject to
$$\begin{aligned} x_1 \qquad\quad &\le 4 \\ 2x_2 &\le 12 \\ 3x_1 + 2x_2 &\le 18 \\ x_1 \ge 0,\ x_2 &\ge 0. \end{aligned}$$

$$\boxed{\begin{aligned} x_1 \qquad\qquad + s_1 \qquad\quad &= 4 \\ 2x_2 \qquad + s_2 \qquad &= 12 \\ 3x_1 + 2x_2 \qquad\qquad + s_3 &= 18 \\ -3x_1 - 5x_2 \qquad\qquad + Z &= 0 \\ x_1, x_2, s_1, s_2, s_3 &\ge 0 \end{aligned}}$$

<u>Standard form</u> $\text{Max } Z = c_1x_1 + \dots + c_nx_n$

subject to
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &\le b_2 \\ &\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &\le b_m \end{aligned}$$

where $\boxed{x_1 \ge 0, x_2 \ge 0, \dots, x_n \ge 0,\ b_1, \dots, b_m \ge 0}$

Ex $\text{Min } C = 2x_1 + 3x_2 + x_3$

s.t.
$$\begin{aligned} x_1 + 4x_2 + 2x_3 &\ge 8 \\ x_1 - 2x_2 + x_3 &\le 6 \Rightarrow -x_1 + 2x_2 - x_3 \ge -6 \\ x_1, x_2, x_3 &\ge 0 \end{aligned}$$

$$A = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 4 & 2 & 8 \\ -1 & 2 & -1 & -6 \\ \hline 2 & 3 & 1 & 1 \end{array}\right] \end{array} \qquad A^T = \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \left[\begin{array}{cc|c} 1 & -1 & 2 \\ 4 & 2 & 3 \\ 2 & -1 & 1 \\ \hline 8 & -6 & 1 \end{array}\right] \end{array}$$

$\text{Max } C' = 8y_1 - 6y_2$

s.t.
$$\begin{aligned} y_1 - y_2 &\le 2 \\ 4y_1 + 2y_2 &\le 3 \\ 2y_1 - y_2 &\le 1 \\ y_1, y_2 &\ge 0 \end{aligned}$$

ตาราง Simplex

	x_1	x_2	s_1	s_2	s_3	
s_1	1	0	1	0	0	4
s_2	0	2	0	1	0	12
s_3	3	2	0	0	1	18
Z	-3	-5	0	0	0	0

$x_1 = 0$
$x_2 = 0$
$s_1 = 4$
$s_2 = 12$
$s_3 = 18$

↓

	x_1	x_2	s_1	s_2	s_3	
s_1	0	0	1	$\frac{1}{3}$	$-\frac{1}{3}$	2
x_2	0	1	0	$\frac{1}{2}$	0	6
x_1	1	0	0	$-\frac{1}{3}$	$+\frac{1}{3}$	2
Z	0	0	0	$\frac{3}{2}$	1	36

$x_1 = 2$
$x_2 = 6$
$s_1 = 2$
$s_2 = 0$
$s_3 = 0$

	y_1	y_2	S_1	S_2	S_3	

The dual problem
Optimal solution is ______________________________ Optimal value is ____________

The original problem
Optimal solution is ______________________________ Optimal value is ____________

2.6 Max and Min with Mixed Problem constraints

Maximization
Constraint equations of the modified problem
Step 1: If any problem constraints have negative constants on the right side, multiply both sides by -1 to obtain a constraint with a nonnegative constant.
Remember to reverse the direction of the inequality if the constraint is an inequality.
Step 2: Add a slack variable for each constraint of the form ≤.
Step 3: Add an *artificial variable* - a *surplus variable* and in each ≥ constraint.
Step 4: Add an artificial variable in each = constraint.
Step 5: For each artificial variable *a*, add the *penalty cost*, $-Ma$ to the objective function. Use the same constant *M* for all artificial variables. Move all variables to the left-hand side.

Example 2.6.1 Find the constraint equations of the modified problem of the following linear programming problem (Do not attempt to solve the problem.)

$$\begin{array}{lrl} \text{Max} & P = & 2x_1 + x_2 - Ma_1 \\ \text{s. t.} & & x_1 + x_2 \le 10 \\ & & -x_1 + x_2 \ge 2 \\ & & x_1 \ge 0, x_2 \ge 0 \end{array}$$

slack variable

surplus variable

artificial variable

$$x_1 + x_2 + s_1 = 10$$

$$-x_1 + x_2 - s_2 + a_1 = 2$$

$$-2x_1 - x_2 + Ma_1 + P = 0$$

$$x_1, x_2, s_1, s_2, a_1 \geq 0$$

ไม่เติม artificial

	x_1	x_2	s_1	s_2	
s_1	1	1	1	0	10
s_2	-1	1	0	-1	2
P	-2	-1	0	0	0

เติม artificial

	x_1	x_2	s_1	s_2	a_1	
s_1	1	1	1	0	0	10
a_1	-1	1	0	-1	1	2
P	-2	-1	0	0	M	0

Example 2.6.2 Find the constraint equations of the modified problem of the following linear programming problem (Do not attempt to solve the problem.).

$$
\begin{array}{llrcrcrl}
\text{Max} & P = & 2x_1 & + & 5x_2 & + & 3x_3 & -Ma_1 - Ma_2 - Ma_3 \\
\text{s. t.} & & x_1 & + & 2x_2 & - & x_3 & \le 7 \\
 & & -x_1 & + & x_2 & - & 2x_3 & \le -5 \Rightarrow x_1 - x_2 + 2x_3 \ge 5 \\
 & & x_1 & + & 4x_2 & + & 3x_3 & \ge 1 \\
 & & 2x_1 & - & x_2 & - & 4x_3 & = 6 \\
 & & & & & & & x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0
\end{array}
$$

$$
\begin{aligned}
x_1 + 2x_2 - x_3 + s_1 &= 7 \\
x_1 - x_2 + 2x_3 - s_2 + a_1 &= 5 \\
x_1 + 4x_2 + 3x_3 - s_3 + a_2 &= 1 \\
2x_1 - x_2 - 4x_3 + a_3 &= 6 \\
-2x_1 - 5x_2 - 3x_3 + Ma_1 + Ma_2 + Ma_3 + P &= 0
\end{aligned}
$$

$x_1, x_2, x_3, s_1, s_2, s_3, a_1, a_2, a_3 \ge 0$, $M > 0$

โดยที่ M ใหญ่มาก ๆ

2.7 Big M Method:

1. Form the preliminary simplex tableau for the modified problem.
2. Use row operations to eliminate the *M*s in the bottom row of the preliminary simplex tableau in the columns corresponding to the artificial variables. The resulting tableau is the initial simplex tableau.
3. Solve the modified problem by applying the simplex method to the initial simplex tableau found in the second step.
4. Relate the optimal solution of the modified problem to the original problem.
   A) if the modified problem has no optimal solution, the original problem has no optimal solution.
   B) if all artificial variables are 0 in the optimal solution to the modified problem, delete the artificial variables to find an optimal solution to the original problem
   C) if any artificial variables are nonzero in the optimal solution, the original problem has no optimal solution.

Example 2.7.1 Solve the following linear programming problem.

$$
\begin{array}{llrcrl}
\text{Max} & P = & 2x_1 & + & x_2 & -Ma_1 \\
\text{s. t.} & & x_1 & + & x_2 & \le 10 \\
 & & -x_1 & + & x_2 & \ge 2 \\
 & & & & & x_1 \ge 0, x_2 \ge 0
\end{array}
$$

$$
\left.
\begin{aligned}
x_1 + x_2 + s_1 &= 10 \\
-x_1 + x_2 - s_2 + a_1 &= 2 \\
-2x_1 - x_2 + Ma_1 + P &= 0 \\
x_1, x_2, s_1, s_2, a_1 &\ge 0
\end{aligned}
\right\}
\text{Modified problem}
$$

	x_1	x_2	x_3	s_1	s_2	a_1	s_3	a_2	a_3	
s_1	1	2	−1	1	0	0	0	0	0	7
a_1	1	−1	2	0	−1	1	0	0	0	5
a_2	1	4	3	0	0	0	−1	1	0	1
a_3	2	−1	−4	0	0	0	0	0	1	6
P	−2	−5	−3	0	0	M	0	M	M	0

		x_1	x_2	S_1	S_2	a_1	
preliminary simplex tableau	S_1	1	1	1	0	0	10
	a_1	−1	1	0	−1	1	2
	P	−2	−1	0	0	M	0
initial simplex tableau	S_1	1	1	1	0	0	10
	a_1	−1	1	0	−1	1	2
$R_3 - MR_2 \to R_3$	P	$-2-M(-1)$ M − 2	$-1-M(1)$ −M − 1 $-(M+1)$	$0-M(0)$ 0	$0-M(-1)$ M	$M-M(1)$ 0	$0-M(2)$ −2M
$R_1 - R_2 \to R_1$	S_1	2	0	1	1	−1	8
	x_2	−1	1	0	−1	1	2
$R_3 + (M+1)R_2 \to R_3$	P	$(M-2)+(M+1)(-1)$ −3	$(-M-1)+(M+1)(1)$ 0	$0+(M+1)(0)$ 0	$M+(M+1)(-1)$ −1	$0+(M+1)(1)$ M+1	$-2M+(M+1)(2)$ 2
$\frac{1}{2}R_1 \to R_1$	x_1	1	0	1/2	1/2	−1/2	4
$R_2 + R_1 \to R_2$	x_2	0	1	1/2	−1/2	1/2	6
$R_3 + 3R_1 \to R_3$	P	0	0	3/2	1/2	M − 1/2	14

$x_1 = 4$, $x_2 = 6$, $S_1 = 0$, $S_2 = 0$, $a_1 = 0$

Optimal solution is $(x_1, x_2) = (4, 6)$ Optimal value is 14

Example 2.7.2 Solve following linear programming problem.

$$
\begin{array}{ll}
\text{Max } P = & x_1 - x_2 + 3x_3 - Ma_1 - Ma_2 \\
\text{s. t.} & x_1 + x_2 \le 20 \\
 & x_1 + x_3 = 5 \\
 & x_2 + x_3 \ge 10 \\
 & x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0
\end{array}
$$

Modified problem

$$
\begin{aligned}
x_1 + x_2 + S_1 &= 20 \\
x_1 + x_3 + a_1 &= 5 \\
x_2 + x_3 - S_2 + a_2 &= 10 \\
-x_1 + x_2 - 3x_3 + Ma_1 + Ma_2 + P &= 0
\end{aligned}
$$

$x_1, x_2, x_3, S_1, S_2, a_1, a_2 \ge 0$,

$M > 0$ เป็นจำนวนจริงใหญ่มาก

		x_1	x_2	x_3	s_1	a_1	s_2	a_2	
preliminary simplex tableau	s_1	1	1	0	1	0	0	0	20
	a_1	1	0	1	0	1	0	0	5
	a_2	0	1	1	0	0	-1	1	10
	P	-1	1	-3	0	M	0	M	0
~~initial simplex tableau~~	s_1	1	1	0	1	0	0	0	20
	a_1	1	0	1	0	1	0	0	5
	a_2	0	1	1	0	0	-1	1	10
$R_4 - MR_2 \to R_4$	P	-1-M(1) -M-1	1-M(0) 1	-3-M(1) -3-M	0-M(0) 0	M-M(1) 0	0-M(0) 0	M-M(0) M	0-M(5) -5M
initial simplex tableau	s_1	1	1	0	1	0	0	0	20
	a_1	1	0	1	0	1	0	0	5
	a_2	0	1	1	0	0	-1	1	10
$R_4 - MR_3 \to R_4$	P	(-M-1)-M(0) -1-M	1-M(1) 1-M	(-3-M)-M(1) -3-2M -(3+2M)	0-M(0) 0	0-M(0) 0	0-M(-1) M	M-M(1) 0	-5M-M(10) -15M
	s_1	1	1	0	1	0	0	0	20
	x_3	1	0	1	0	1	0	0	5
$R_3 - R_3 \to R_3$	a_3	-1	1	0	0	-1	-1	1	5
$R_4 + (3+2M)R_2 \to R_4$	P	2+M	1-M	0	0	3+2M	M	0	-5M+15
$R_1 - R_3 \to R_1$	s_1	2	0	0	1	1	1	-1	15
	x_3	1	0	1	0	1	0	0	5
	x_2	-1	1	0	0	-1	-1	1	5
$R_4 - (1-M)R_3 \to R_4$	P	3	0	0	0	4+M	1	-1+M	10

$x_1 = 0$
$x_2 = 5$
$x_3 = 5$
$s_1 = 15$
$s_2 = 0$
$a_1 = 0$
$a_2 = 0$

Optimal solution is $(x_1, x_2, x_3) = (0, 5, 5)$ Optimal value is 10

Minimization

1. Change the objective function from minimization to maximization by multiplying -1.
2. Use simplex method for maximization to solve the problem.
3. The solution of the maximization version of the problem is the same as the original minimization problem.

If the objective function value of the maximization version is z, the objective function value of the minimization version is -z.

Example 2.7.3 Solve following linear programming problem.

$$\begin{array}{llcl} \text{Min} & C = 30x_1 + 30x_2 + 10x_3 \\ \text{s. t.} & x_1 + x_2 + x_3 \geq 6 \\ & 2x_1 + x_2 + 2x_3 \leq 10 \\ & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \end{array}$$

$$\begin{array}{ll} \text{Max} & C' = -30x_1 - 30x_2 - 10x_3 - Ma_1 \\ \text{s.t.} & x_1 + x_2 + x_3 \geqslant 6 \\ & 2x_1 + x_2 + 2x_3 \leqslant 10 \\ & x_1, x_2, x_3 \geqslant 0 \end{array}$$

Modified problem

$$\left\{ \begin{array}{l} x_1 + x_2 + x_3 - S_1 + a_1 = 6 \\ 2x_1 + x_2 + 2x_3 + S_2 = 10 \\ 30x_1 + 30x_2 + 10x_3 + Ma_1 + P = 0 \\ x_1 \geqslant 0, x_2, x_3, S_1, S_2, a_1 \geqslant 0, M > 0 \end{array} \right.$$

		-100

Optimal solution is $(x_1, x_2, x_3) = (0, 2, 4)$ Optimal value is $C' = -100$ (Max)
$C = 100$ (Min)