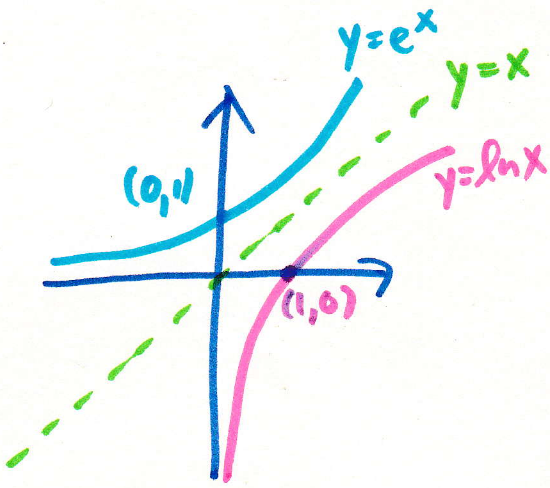
Warm-up problem

$$\lim_{x\to\infty} x^2 - 2x = \lim_{x\to\infty} x^2\left(1-\frac{2}{x}\right) = +\infty$$

$$\lim_{x\to-\infty}(x^2-2x) = +\infty$$

$$\lim_{x\to\infty}\frac{1}{x^2-2x} = 0$$

$$\lim_{x\to0^+}\left(\frac{1}{x^2}+\frac{1}{x}\right) = +\infty \quad \Bigg| \quad \lim_{x\to0^+}\left(\overset{\infty}{\frac{1}{x^2}} - \overset{\infty}{\frac{1}{x}}\right) = \lim_{x\to0^+}\left(\frac{\overset{1}{1-x}}{x^2}\right) = +\infty$$



$$\lim_{x\to0^+}(x-1)\ln x = +\infty$$

$$\lim_{x\to0^+}\left(\frac{1}{x}-\frac{2x+1}{x^2+x}\right) = \lim_{x\to0^+}\frac{(x+1)-(2x+1)}{x^2+x} = \lim_{x\to0^+} -\frac{x}{x^2+x}$$

$x(x+1)$

$$= \lim_{x\to0^+} -\frac{\cancel{x}}{\cancel{x}(x+1)}$$

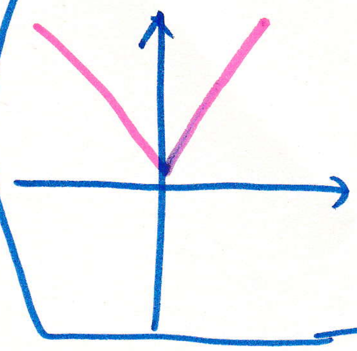
$$= \lim_{x\to0^+} -\frac{1}{x+1}$$

$$= -1 \qquad \#$$

Limits involving radicals

$$\sqrt{x^2} = |x|$$

$$f(x) = \begin{cases} x & ; x \geq 0 \\ -x & ; x < 0 \end{cases}$$



Example 24 Find the following limits.

1. $\lim\limits_{x\to-\infty} \dfrac{\sqrt{4x^2+3x+2}}{2x-7}$.

$$\cancel{\lim_{x\to-\infty} \frac{\frac{\sqrt{4x^2+3x+2}}{x}}{\frac{2x}{x} - \frac{7}{x}}}$$

$$= \lim_{x\to-\infty} \frac{\frac{\sqrt{4x^2+3x+2}}{|x|}}{\frac{2x-7}{|x|}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{\frac{4x^2}{x^2}+\frac{3x}{x^2}+\frac{2}{x^2}}}{\frac{2x-7}{|x|}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{\frac{4x^2}{x^2}+\frac{3x}{x^2}+\frac{2}{x^2}}}{\frac{2x-7}{-x}}$$

$$= \lim_{x\to-\infty} \frac{\sqrt{4+\frac{3}{x}\nearrow 0+\frac{2}{x^2}\nearrow 0}}{-2+\frac{7}{x}\nearrow 0}$$

$$= \frac{\sqrt{4}}{(-2)} = -1$$

Ex $\lim\limits_{x\to+\infty} \dfrac{\sqrt{x^2+2}}{3x-6} = \lim\limits_{x\to+\infty} \dfrac{\frac{\sqrt{x^2+2}}{|x|}}{\frac{3x-6}{|x|}}$

$$= \lim_{x\to+\infty} \frac{\sqrt{\frac{x^2}{x^2}+\frac{2}{x^2}}}{\frac{3x-6}{x}}$$

$$= \lim_{x\to+\infty} \frac{\sqrt{1+\frac{2}{x^2}\nearrow 0}}{3-\frac{6}{x}\nearrow 0}$$

$$= \frac{\sqrt{1}}{3} = \frac{1}{3} \quad \#$$

2. $\lim_{x\to+\infty}\left(x^2-\sqrt{x^4+9}\right)$. $= \lim_{x\to+\infty} (x^2-\sqrt{x^4+9})\dfrac{(x^2+\sqrt{x^4+9})}{(x^2+\sqrt{x^4+9})}$

$$= \lim_{x\to+\infty} \frac{\cancel{x^4}-(\cancel{x^4}+9)}{x^2+\sqrt{x^4+9}}$$

$$= \lim_{x\to+\infty} -\frac{9}{x^2+\sqrt{x^4+9}}$$

$$= \lim_{x\to\infty} -\frac{\frac{9}{x^2}}{\frac{x^2}{x^2}+\frac{\sqrt{x^4+9}}{x^2}}$$

$$= \lim_{x\to+\infty} -\frac{\cancel{\frac{9}{x^2}}\to 0}{1+\sqrt{\cancel{\frac{x^4}{x^4}}+\cancel{\frac{9}{x^4}}\to 0}}$$

$$= 0 \qquad \#$$

$$\lim_{x\to 0}\frac{3x-\sin x}{x} = ?$$

3.6 Indeterminate Forms

$\frac{0}{0}, \frac{\infty}{\infty}, \infty - \infty, 0 \cdot \infty, 1^\infty, \infty^0, 0^0$

3.6.1 L'Hospital's Rule

THEOREM 3.4 Let f, g be differentiable functions on an open interval containing $x = a$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Diff quotients

$$\boxed{\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}}$$

Note that the theorem also works for one-sided limits.

Example 25 Find $\lim_{x\to 0} \frac{\boxed{3x - \sin x}}{\boxed{x}}$.

$$\boxed{\lim_{x\to 0} 3x - \sin x = 0}, \quad \boxed{\lim_{x\to 0} x = 0}$$

$$\lim_{x\to 0} \frac{3x - \sin x}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}(3x - \sin x)}{\frac{d}{dx}(x)}$$

$$= \lim_{x\to 0} \frac{3 - \cos x}{1}$$

$$= 3 - 1 = 2 \qquad \#$$

ダメ

$$\cancel{\lim_{x\to 0} \frac{x+6}{x+2} = \lim_{x\to 0} \frac{1}{1} = 1}$$

$$\boxed{\lim_{x\to 0} \frac{x+6}{x+2} = \frac{6}{2} = 3} \checkmark$$

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = (2+2) = 4$$

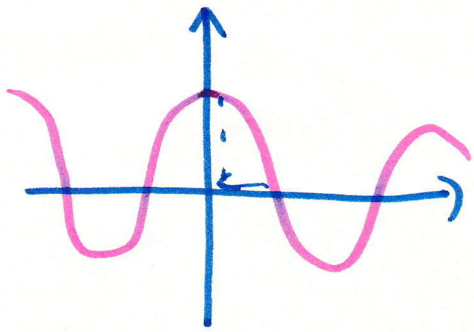
<u>L'Hôpital</u>

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{2x}{1} = 4 \qquad \#$$

Example Compute

$$\lim_{x \to 0^+} \frac{2+x}{1-\cos x}$$

$$\lim_{x \to 0^+} (2+x) = 2 \quad , \quad \lim_{x \to 0^+} (1-\cos x) = 0$$



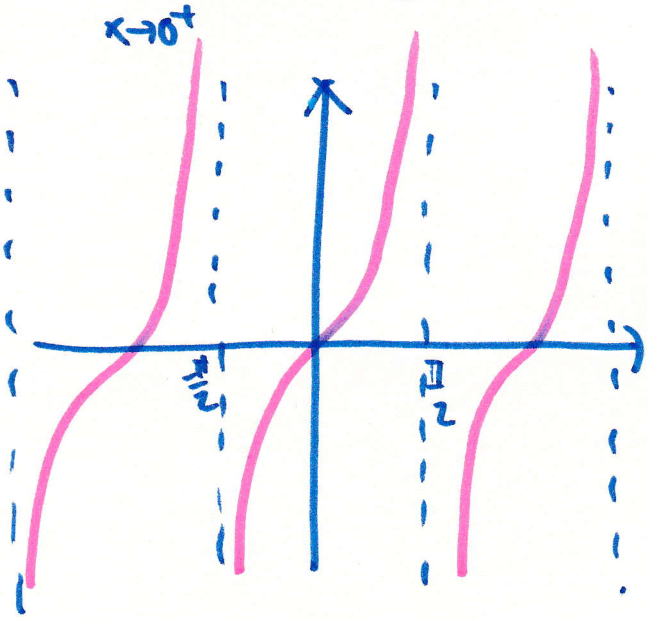
$$\therefore \lim_{x \to 0^+} \frac{2+x}{1-\cos x} = +\infty$$

Example Compute

$$\lim_{x \to 0^+} \frac{\tan x}{x^2}$$

$$\lim_{x \to 0^+} \frac{\tan x}{x^2} = \lim_{x \to 0^+} \frac{\sec^2 x}{2x} = +\infty$$

$$\lim_{x \to 0^+} \tan x = 0$$



Theorem 3.5 Let f, g be differentiable functions on an open interval containing $x = a$ and $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \pm\infty$. If $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}.$$

Note that the theorem also works for one-sided limits.

Example 26 Find $\lim_{x\to\infty} \frac{\boxed{2x+3}}{\boxed{5x+7}}$.

$$\boxed{\lim_{x\to\infty} (2X+3) = +\infty} \quad , \quad \boxed{\lim_{x\to\infty} (5X+7) = +\infty}$$

$$\lim_{x\to\infty} \frac{2X+3}{5X+7} = \lim_{x\to\infty} \frac{\frac{d}{dx}(2X+3)}{\frac{d}{dx}(5X+7)}$$

$$= \lim_{x\to\infty} \frac{2}{5} = \frac{2}{5}$$

#

$A \cdot B = A \cdot \dfrac{1}{\frac{1}{B}}$

แทน $\boxed{B \cdot \dfrac{1}{\frac{1}{A}}}$

Other Indeterminate forms

Solving other indeterminate forms require us to change the limit so that we can use L'hospital's rule.

$0 \cdot \infty$

Example 27 Find $\lim_{x\to\infty} x \tan \frac{1}{x}$.

เมื่อ $x \to \infty$, $\lim_{x\to\infty} x = +\infty$

เมื่อ $x \to \infty$, $\frac{1}{x} \to 0$ จะได้ว่า $\tan\left(\frac{1}{x}\right) \to 0$

$$\lim_{x\to\infty} x \cdot \tan\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{x}{\frac{1}{\tan\left(\frac{1}{x}\right)}} = \lim_{x\to\infty} \frac{x}{\cot\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{-\operatorname{cosec}^2\left(\frac{1}{x}\right)\frac{d}{dx}\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{1}{-\operatorname{cosec}^2\left(\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{x^2}{\operatorname{cosec}^2\left(\frac{1}{x}\right)}$$

$\vdots$

$$\lim_{x\to\infty} x \cdot \tan\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{\tan\left(\frac{1}{x}\right)}{\frac{1}{x}} = \lim_{x\to\infty} \frac{\sec^2\left(\frac{1}{x}\right)\frac{d}{dx}\left(\frac{1}{x}\right)}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to\infty} \frac{\left[\sec^2\left(\frac{1}{x}\right)\right]\cancel{\left(-\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}$$

$$= \lim_{x\to\infty} \sec^2\left(\frac{1}{x}\right)$$

เมื่อ $x \to \infty$, $\frac{1}{x} \to 0$ $\quad \sec\left(\frac{1}{x}\right) \to 1$

$\therefore \sec^2\left(\frac{1}{x}\right) \to 1$

$= 1$ #

$\infty - \infty$

Example 28 Find $\displaystyle\lim_{x\to 0^+}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

$$\lim_{x\to 0^+}\left(\frac{1}{\sin x} - \frac{1}{x}\right) = \lim_{x\to 0^+}\left(\frac{x-\sin x}{x\sin x}\right)$$

L'Hôpital
1 รอบ

$$= \lim_{x\to 0^+} \boxed{\frac{1-\cos x}{x\cos x+\sin x}}$$

diff
แยกกัน

$$= \lim_{x\to 0^+} \frac{\frac{d}{dx}[1-\cos x]}{\frac{d}{dx}[x\cos x+\sin x]}$$

$$= \lim_{x\to 0^+} \frac{\sin x}{[x(-\sin x)+\cos x+\cos x]}$$

$$= \lim_{x\to 0^+} \frac{\sin x}{-x\sin x+2\cos x}$$

$$= 0 \qquad \#$$

In case the limit of the form $0^0, \infty^0, 1^\infty$, apply logarithm to the limit.

Example 29 Find $\lim\limits_{x\to 0^+} (\sin x)^x$.

สมมติว่า $y = (\sin x)^x$

take ln ทั้งสองข้าง

$$\ln y = \ln(\sin x)^x$$

$$\ln y = x\ln(\sin x)$$

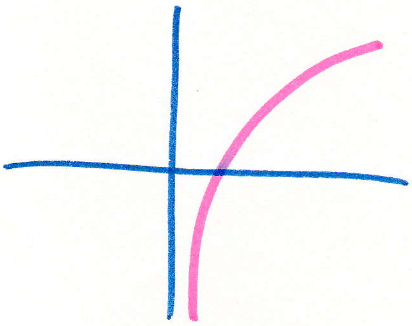
$$\lim_{x\to 0^+} \ln y = \boxed{\lim_{x\to 0^+} x\ln(\sin x)}$$

$0\cdot\infty$

พิจารณา $\boxed{\lim\limits_{x\to 0^+} x\ln(\sin x)} = \lim\limits_{x\to 0^+} \dfrac{\ln(\sin x)}{\frac{1}{x}}$

$x \to 0^+$, $\sin x \to 0^+$

$\ln(\sin x) \to -\infty$



$$= \lim_{x\to 0^+} \frac{\frac{d}{dx}[\ln(\sin x)]}{\frac{d}{dx}\left(\frac{1}{x}\right)}$$

$$= \lim_{x\to 0^+} \frac{\frac{1}{\sin x}\frac{d}{dx}(\sin x)}{\left(-\frac{1}{x^2}\right)}$$

$$= \lim_{x\to 0^+} -\frac{\frac{\cos x}{\sin x}}{\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} -\frac{\cot x}{\frac{1}{x^2}}$$

$$= \lim_{x\to 0^+} -\frac{x^2}{\tan x}$$

$$= \lim_{x\to 0^+} -\frac{\frac{d}{dx}[x^2]}{\frac{d}{dx}[\tan x]}$$

$$= \lim_{x\to 0^+} -\frac{2x}{\sec^2 x} \boxed{= 0}$$

$$\therefore \lim_{x\to 0^+} \ln y = 0$$

$$\left(y = (\sin x)^x \right)$$

$$\lim_{x \to 0^+} \ln y = 0$$

ถ้า $\ln y = 0$

$$y = e^0 = 1$$

$$\boxed{e^{\ln y} = y}$$

① ฟังก์ชัน $\boxed{f(x) = e^x}$ ต่อเนื่องที่ทุกจุดใน จ.น.จริง)

② $\lim_{x \to 0^+} \ln \boxed{y} = 0$ $\left(y = (\sin x)^x\right)$

Thm 1.7 If $\lim_{x \to c} g(x) = L$ and if f is cont. at L,

then
$$\lim_{x \to c} f(g(x)) = f\left(\lim_{x \to c} g(x)\right)$$

จาก Thm 1.7 ของ $f(x) = e^x$

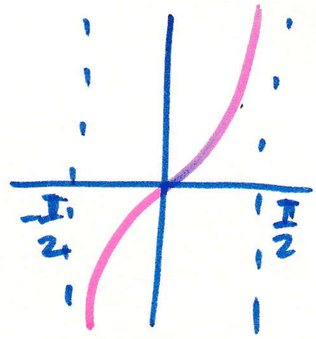
$$\lim_{x \to 0^+} \boxed{e^{\ln (\sin x)^x}} = e^{\boxed{\lim_{x \to 0^+} \ln (\sin x)^x}} \quad = 0$$

ดังนั้น $\lim_{x \to 0^+} (\sin x)^x = e^0 = 1$ #.

Failures of L'Hospital's Rule

Sometimes using L'hospital's Rule will not solve the problem as in following example.

Example 30 Find $\displaystyle\lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan x}$.



$$\lim_{x\to(\frac{\pi}{2})^-} \frac{\sec x}{\tan x} = \lim_{x\to(\frac{\pi}{2})^-} \frac{\frac{d}{dx}[\sec x]}{\frac{d}{dx}[\tan x]}$$

$$= \lim_{x\to(\frac{\pi}{2})^-} \frac{\cancel{\sec} x \tan x}{\sec^{\cancel{2}} x}$$

$$= \lim_{x\to(\frac{\pi}{2})^-} \frac{\tan x}{\sec x}$$

$$= \lim_{x\to(\frac{\pi}{2})^-} \frac{\cancel{\sec^2} x}{\cancel{\sec x} \tan x} = \lim_{x\to(\frac{\pi}{2})^-} \frac{\sec x}{\tan x}$$

$$\lim_{x\to(\frac{\pi}{2})^-} \frac{\sec x}{\tan x} = \lim_{x\to(\frac{\pi}{2})^-} \frac{1/\cancel{\cos x}}{\frac{\sin x}{\cancel{\cos x}}} = \lim_{x\to(\frac{\pi}{2})^-} \frac{1}{\sin x} = 1$$