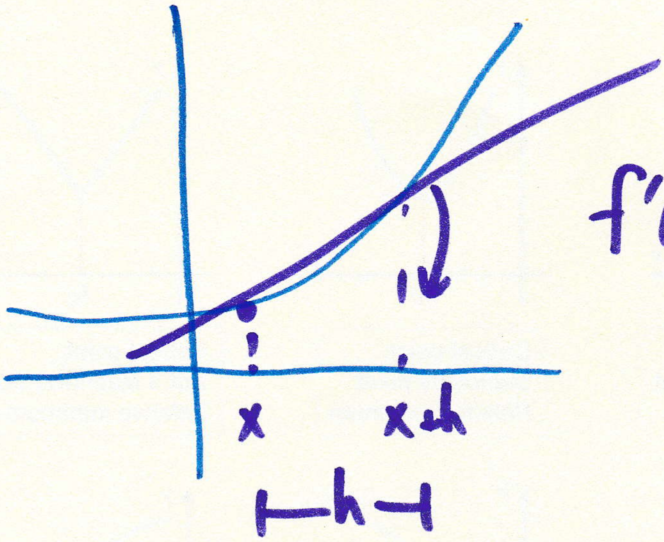
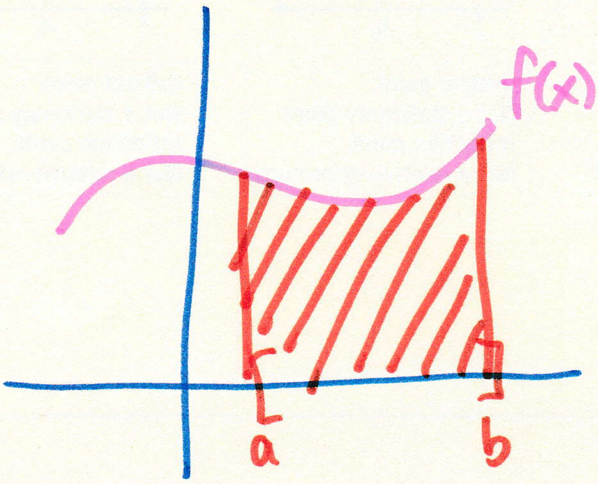


$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

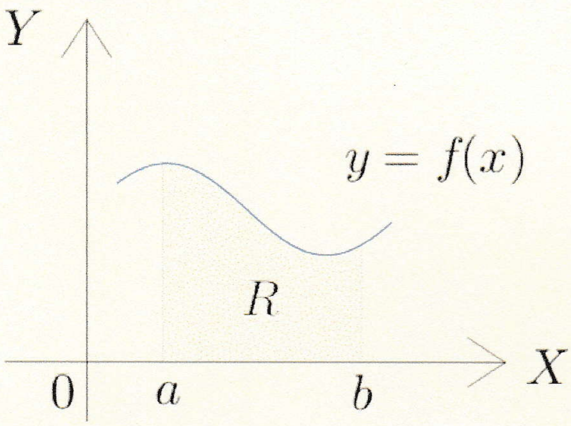


$f(x)\geq 0$

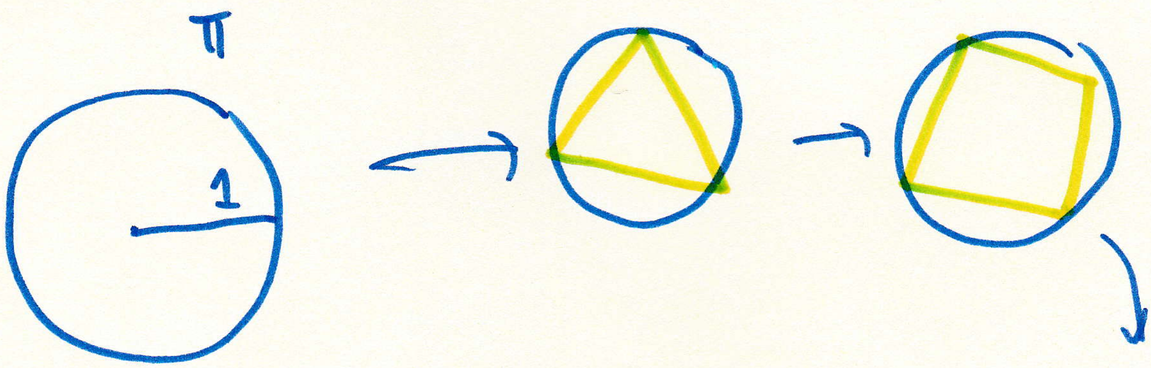
5 Integration

5.1 An Overview of the Area Problem

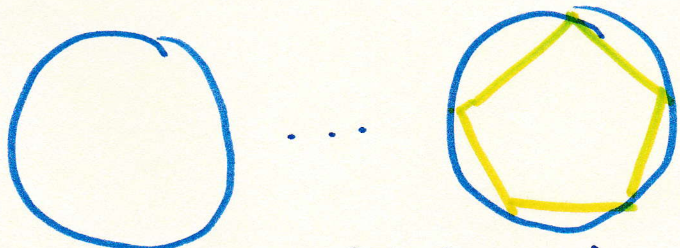
> Given a function f that is continuous and nonnegative on an interval $[a, b]$, find the area between the graph of f and the interval $[a, b]$ on the x-axis (Figure 5.1).



Archimedes' method of exhaustion



ต้องการหา พ.ท. วงกลม

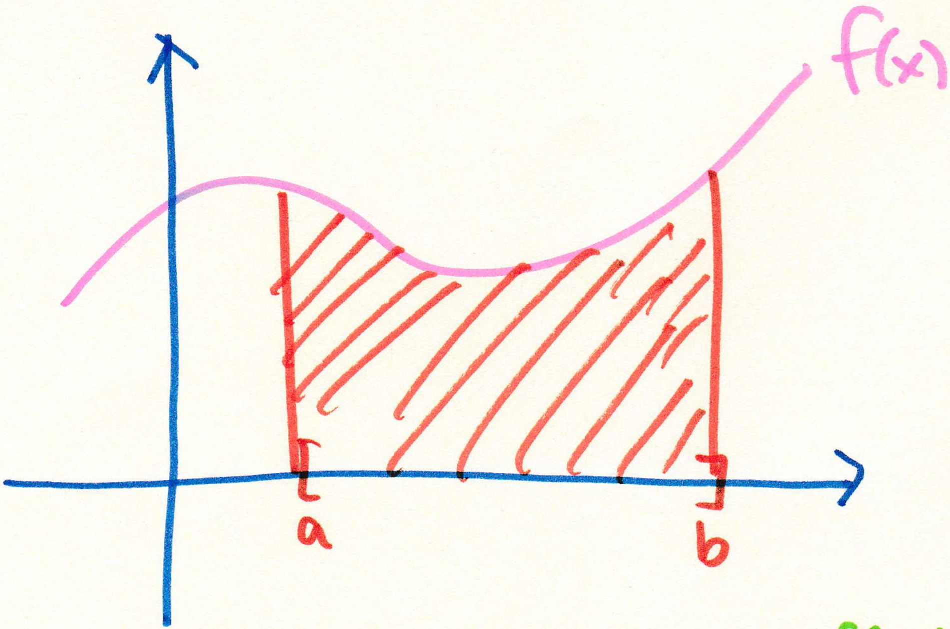


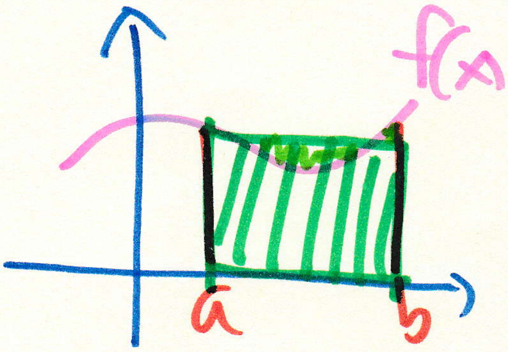
regular n-gon A_n : พื้นที่รูป n เหลี่ยมด้านเท่

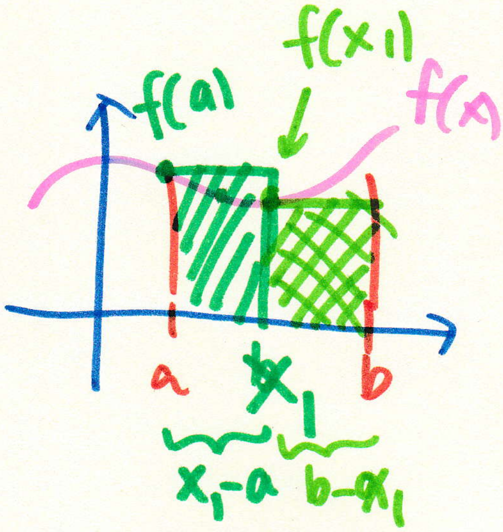
$$\boxed{A = \lim_{n \to \infty} A_n}$$

$f(x) > 0$

Let $y = f(x)$ be a function over the interval $[a, b]$. We consider the rectangle method for computing the area A under the curve $f(x)$ on the interval $[a, b]$.

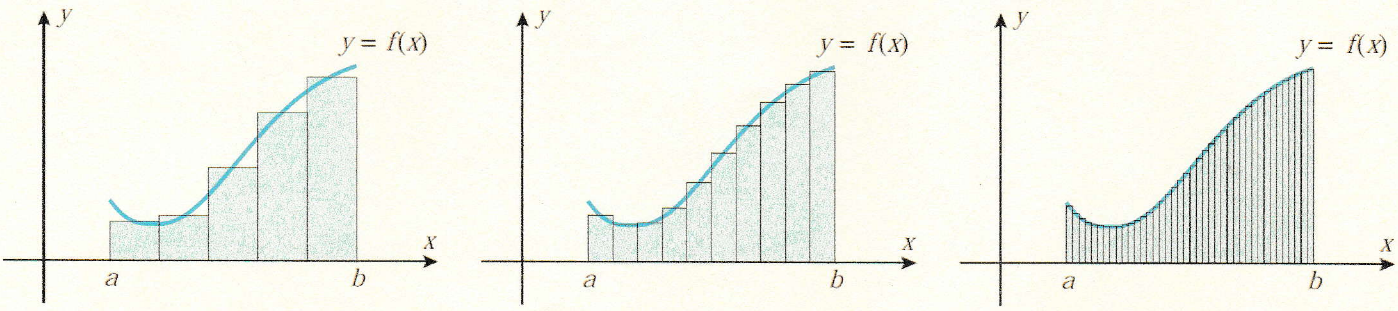






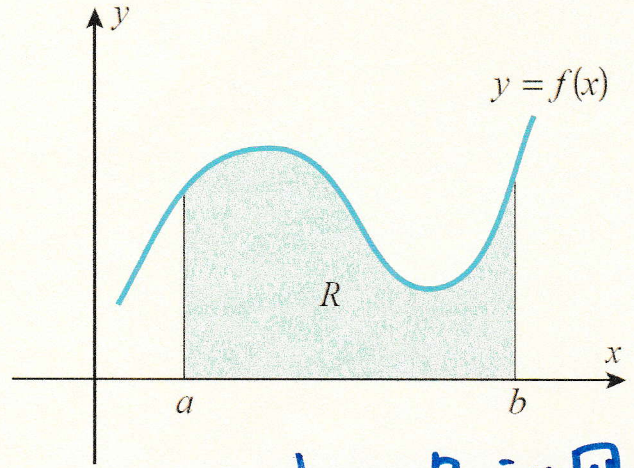
That is, if A denotes the exact area under the curve and A_n denotes the approximation to A using n rectangles, then

$$A = \lim_{n \to \infty} A_n$$

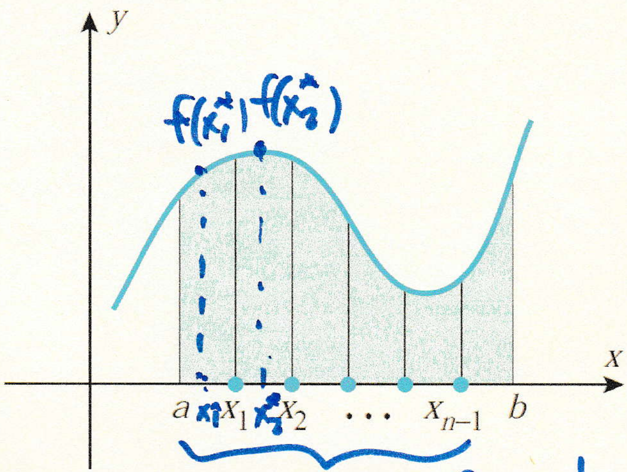


DEFINITION 5.1 (Area Under a Curve) If the function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the area A under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$\boxed{A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.}$$



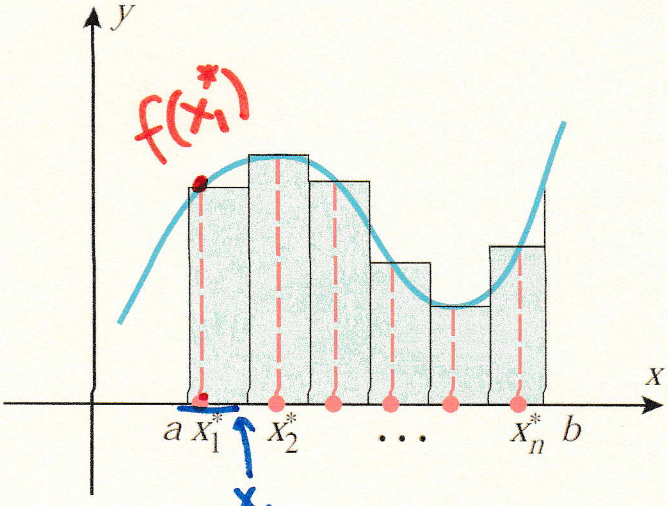
ต้องการประมาณ R ด้วย ☐ n ☐.

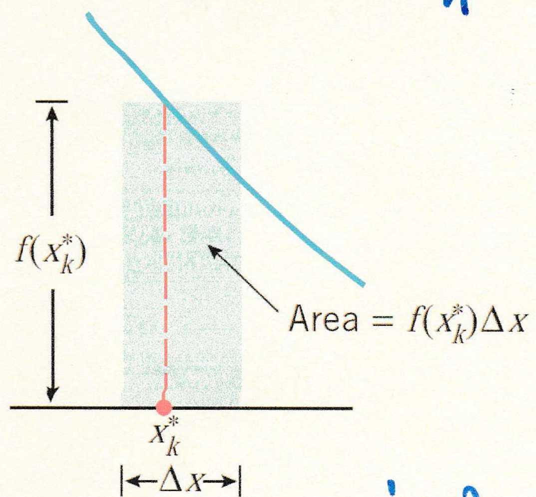


แบ่ง $[a,b]$ ออกเป็น n ช่วง

สมมติว่า แต่ละช่วงยาวเท่ากัน

$\therefore$ แต่ละช่วงยาว $\dfrac{b-a}{n} = \Delta x$



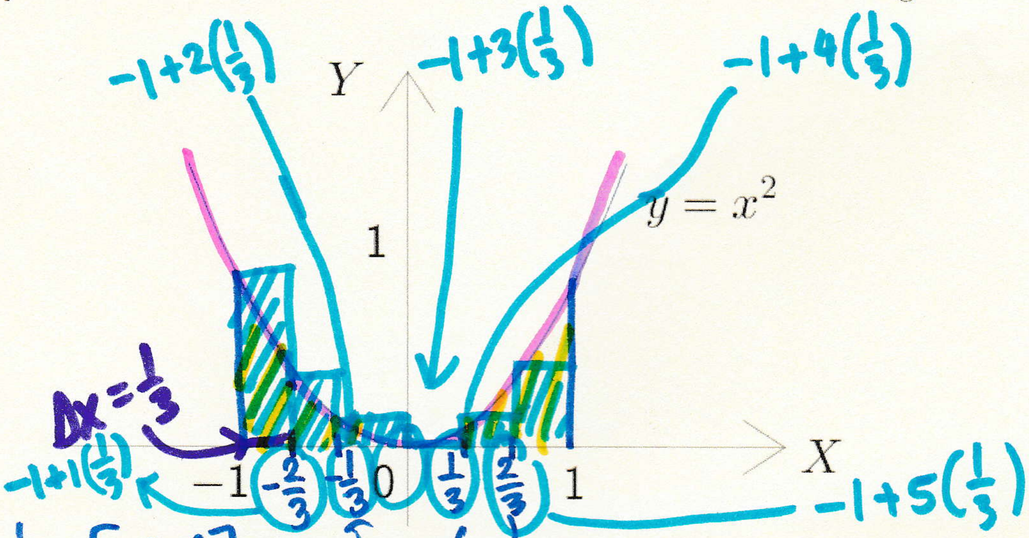


ไม่ว่าที่ k ใดๆ

$$A_n = f(x_1^*)\Delta x + f(x_2^*)\Delta x + f(x_3^*)\Delta x + \dots + f(x_n^*)\Delta x$$

$$A_n = \sum_{k=1}^{n} f(x_k^*)\Delta x \Rightarrow \boxed{A = \lim \sum_{k=1}^{n} f(x_k^*)\Delta x}$$

It is probably easiest to see how we do this with an example. So let's determine the area between $f(x) = x^2$ on $[-1, 1]$. In other words, we want to determine the area of the shaded region below.



Step 1 แบ่ง $[-1,1]$ ออกเป็น 6 ส่วน

$\therefore$ ความกว้างของแต่ละช่วง $\Rightarrow \Delta x = \frac{1-(-1)}{6} = \frac{1}{3}$

$$A_6 = f(-1)(\tfrac{1}{3}) + f(-\tfrac{2}{3})(\tfrac{1}{3}) + f(-\tfrac{1}{3})(\tfrac{1}{3}) + f(0)(\tfrac{1}{3}) + f(\tfrac{1}{3})(\tfrac{1}{3}) + f(\tfrac{2}{3})(\tfrac{1}{3})$$

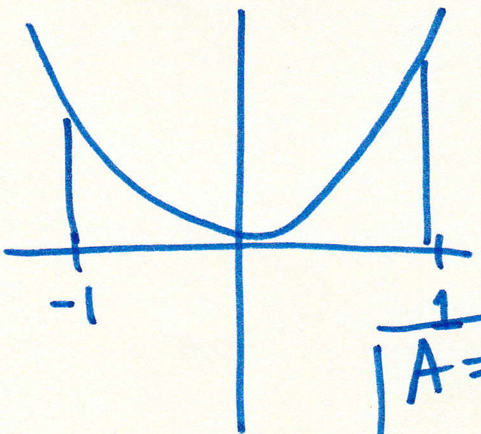
$$= f(-1+0(\tfrac{1}{3}))(\tfrac{1}{3}) + f(-1+1(\tfrac{1}{3}))(\tfrac{1}{3}) + f(-1+2(\tfrac{1}{3}))(\tfrac{1}{3}) +$$

$$f(-1+3(\tfrac{1}{3}))(\tfrac{1}{3}) + f(-1+4(\tfrac{1}{3}))(\tfrac{1}{3}) + f(-1+5(\tfrac{1}{3}))(\tfrac{1}{3})$$

$$= \sum_{k=0}^{5} f(-1+k(\tfrac{1}{3}))(\tfrac{1}{3})$$

$$A_6 = \frac{1}{3}\sum_{k=0}^{5}\left(-1+\frac{k}{3}\right)^2$$

แบ่ง n ชิ้น แต่ละชิ้นกว้าง $\Delta x = \frac{1-(-1)}{n} = \frac{2}{n}$



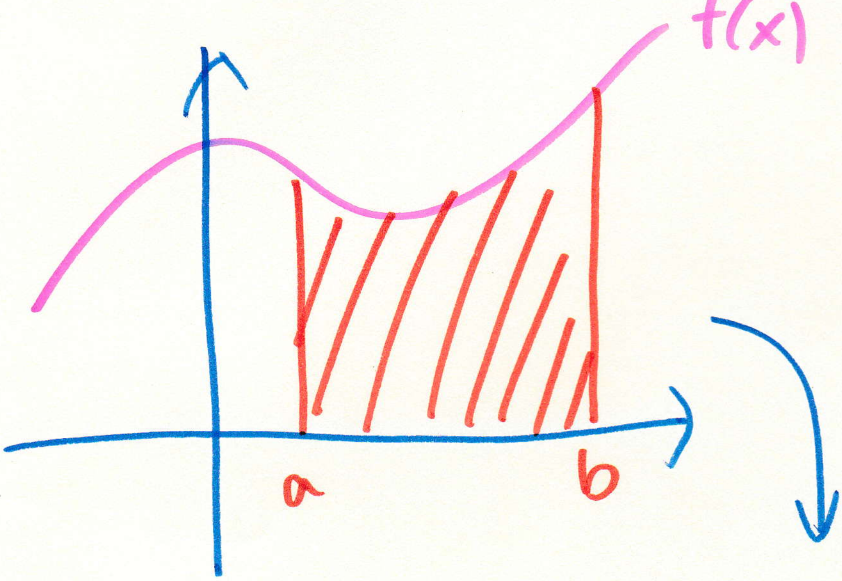
$$A_n = \sum_{k=0}^{n-1} \underbrace{\left(-1+k\cdot\frac{2}{n}\right)^2}_{f(-1+k\frac{2}{n})} \underbrace{\left(\frac{2}{n}\right)}_{\text{ความกว้าง } \Delta x}$$

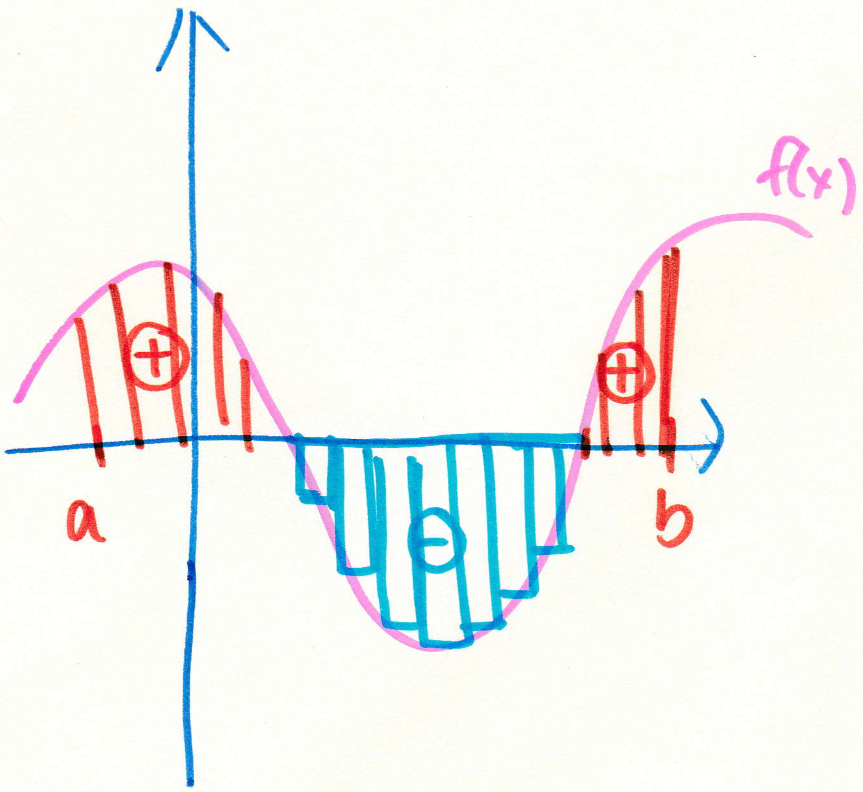
$$\boxed{A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{k=0}^{n-1}\left(-1+k\cdot\frac{2}{n}\right)^2\cdot\left(\frac{2}{n}\right)}$$

ปัญหาท้าทายปัญญา

แสดงว่า

$$A = \lim_{n\to\infty} A_n = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(-1 + k\cdot\frac{2}{n}\right)^2 \cdot \left(\frac{2}{n}\right) = \frac{2}{3}$$





$$1+2+3+\ldots+n$$

$$= \sum_{K=1}^{n} K$$

$$\underbrace{1^2+2^2+3^2+\ldots+n^2} = \sum_{k=1}^{n} k^2$$

$$= \sum_{k=0}^{n-1} (k+1)^2$$

Example 5.1

$\sum_{k=4}^{8} k^3 =$

$\sum_{k=0}^{5} (-1)^k (2k-1) =$

5.2.2 Properties of Sums

Theorem 5.1

(a) $\sum_{k=1}^{n} c a_k = c \sum_{k=1}^{n} a_k$

(b) $\sum_{k=1}^{n} (a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(c) $\sum_{k=1}^{n} (a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

$\sum_{k=1}^{n} c = nc$

$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Table 5.1 below shows the result of evaluating (5.1) on a computer for some increasingly large values of n. These computations suggest that the exact area is close to ....

n	6	10	100	1,000	10,000
A_n	0.7	0.68	0.6668	0.666668	0.66666668

Table 5.1: estimation of area

So, increasing the number of rectangles improves the accuracy of the estimation as we would guess. Later in this chapter we will show that

$$\lim_{n\to\infty} A_n = \frac{2}{3}.$$

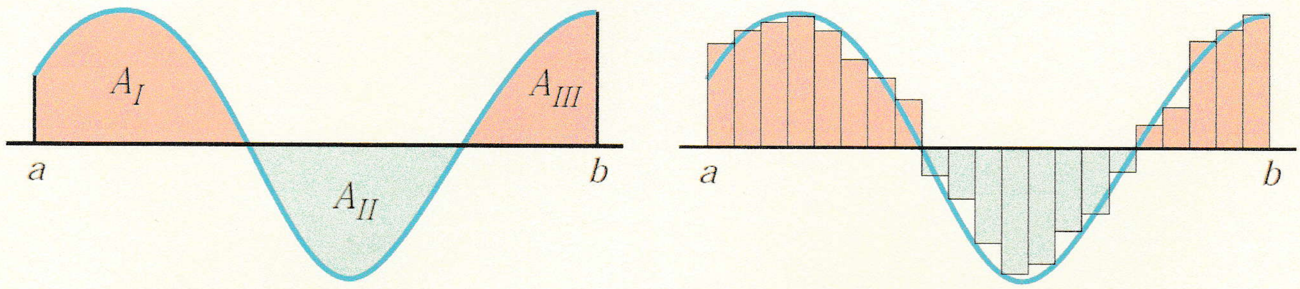
5.2.5 Net Signed Area

If f is continuous and attains both positive and negative values on $[a, b]$, then the limit

$$\lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

no longer represents the area between the curve $y = f(x)$ and the interval $[a, b]$ on the x-axis; rather, it represents a difference of areas — the area of the region that is above the interval $[a, b]$ and below the curve $y = f(x)$ minus the area of the region that is below the interval $[a, b]$ and above the curve $y = f(x)$. We call this the **net signed area**.

f(x)



(a) (b)

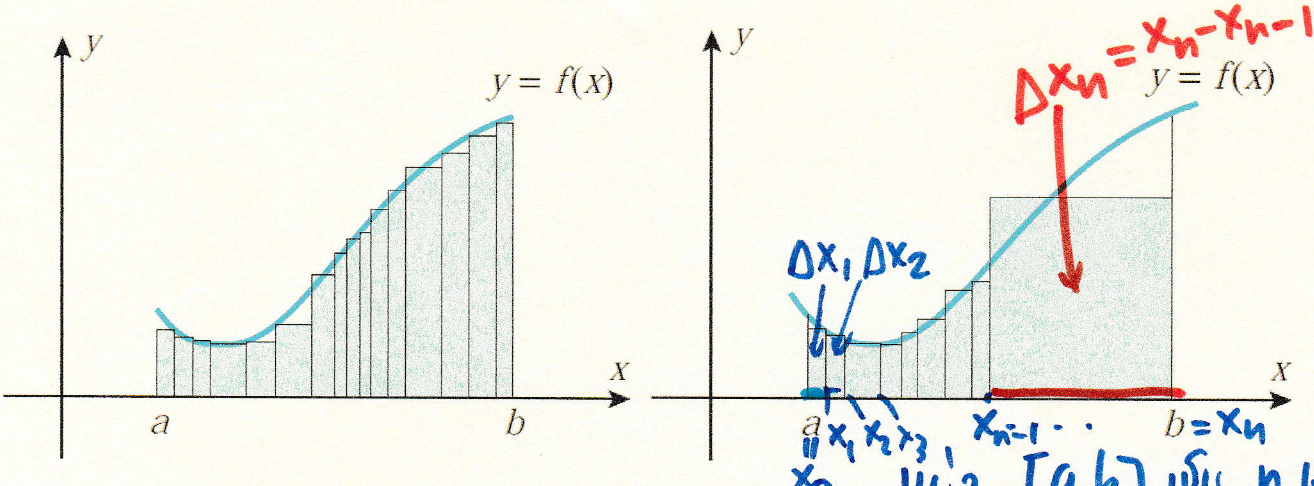
For example, in Figure 5.5, the net signed area between the curve $y = f(x)$ and the interval $[a, b]$ is

$$(AI + AIII) - AII = [\text{ area above } [a, b]] - [\text{ area below } [a, b]]$$

DEFINITION 5.2 (Net Signed Area) If the function f is continuous on $[a, b]$, then the net signed area A between $y = f(x)$ and the interval $[a, b]$ is defined by

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x.$$

Definite Integral



แบ่ง $[a,b]$ เป็น n ช่วงย่อย

A partition of the interval $[a, b]$ is a collection of points

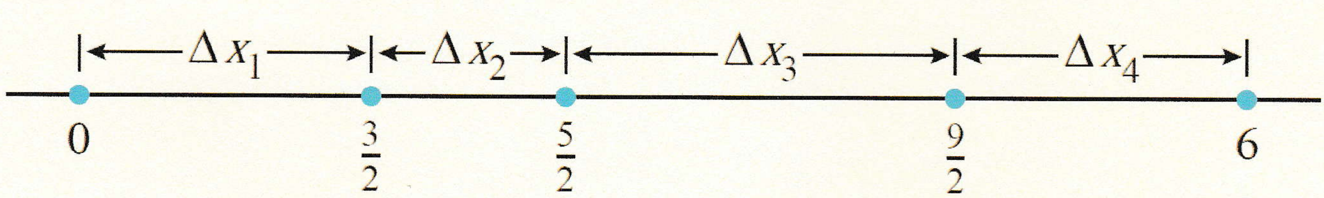
$$a = x_0 < x_1 < x_2 < \cdots < x_{n1} < x_n = b$$

that divides $[a, b]$ into n subintervals of lengths

$$\Delta x_1 = x_1 - x_0,\ \Delta x_2 = x_2 - x_1,\ \Delta x_3 = x_3 - x_2,\ \ldots,\ \Delta x_n = x_n - x_{n-1}$$

The partition is said to be regular provided the subintervals all have the same length

$$\Delta x_k = \Delta x = \frac{b-a}{n}.$$



$$\boxed{\max \Delta x_k = \Delta x_3 = \frac{9}{2} - \frac{5}{2} = 2}$$

เมื่อยิ่งให้ความกว้างช่วงที่มากที่สุด เข้าใกล้ 0

จะมีค่าเหมือนกับ กรณีแบ่งที่ $n \to \infty$

$$A = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$

If we are to generalize Definition 5.2.4 so that it allows for unequal subinterval widths, we must replace the constant length Δx by the variable length Δx_k. When this is done the sum

$$\sum_{k=1}^{n} f(x_k^*)\Delta x \text{ is replaced by } \sum_{k=1}^{n} f(x_k^*)\Delta x_k.$$

We also need to replace the expression $n\infty$ by an expression that guarantees us that the lengths of all subintervals approach zero. We will use the expression $\max \Delta x_k \to 0$ for this purpose.

DEFINITION A function f is said to be integrable on a finite closed interval $[a, b]$ if the limit

$$\boxed{\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k}$$

exists and does not depend on the choice of partitions or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol

Upper limit

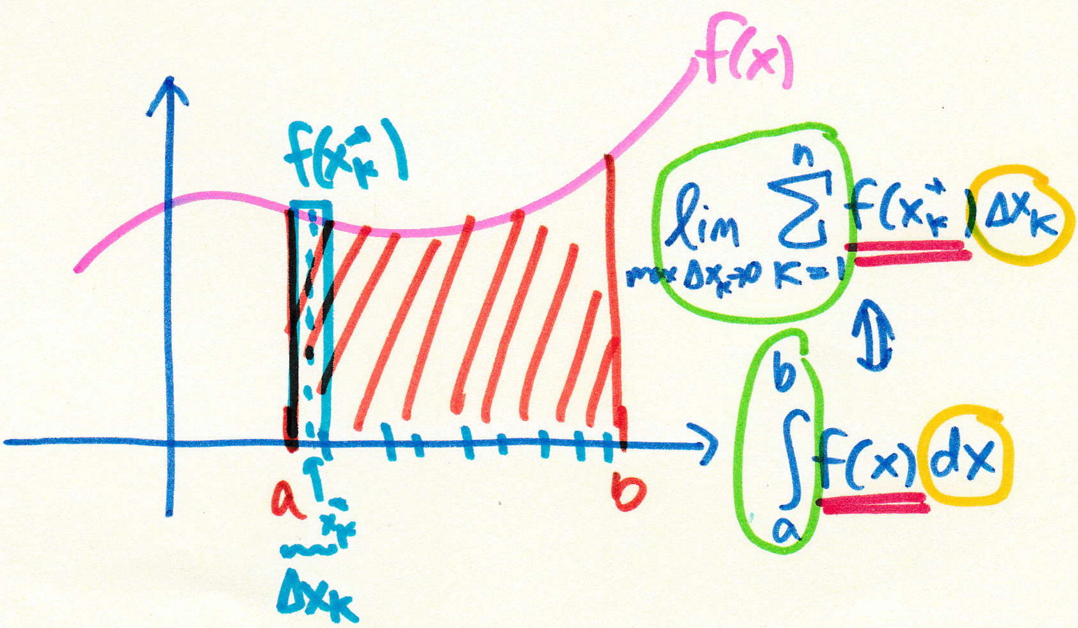
integrand

$$\boxed{\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k.}$$

lower limit

differential

which is called the *definite integral* of f from a to b. The numbers a and b are called the *lower limit of integration* and the *upper limit of integration* , respectively, and $f(x)$ is called the *integrand* .



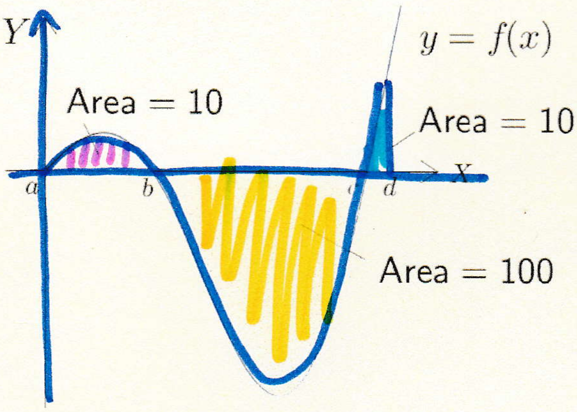
Theorem 5.2 If a function f is continuous on an interval $[a,b]$, then f is integrable on $[a,b]$, and the net signed area A between the graph of f and the interval $[a,b]$ is

$$A = \int_a^b f(x)dx.$$

Example 5.2 Use the areas shown in the figure to find

(a) $\int_a^b f(x)dx$ (b) $\int_b^c f(x)dx$ (c) $\int_a^c f(x)dx$ (d) $\int_a^d f(x)dx$

Solution



(a) $\int_a^b f(x)dx = 10$

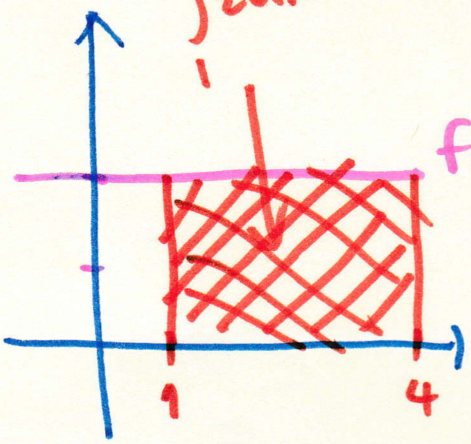
(b) $\int_b^c f(x)dx = -100$

(c) $\int_a^c f(x)dx$ $= 10 - (100)$
$= -90$

(d) $\int_a^d f(x)dx$ $= (10+10) - 100$
$= -80$

Example 5.3 Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

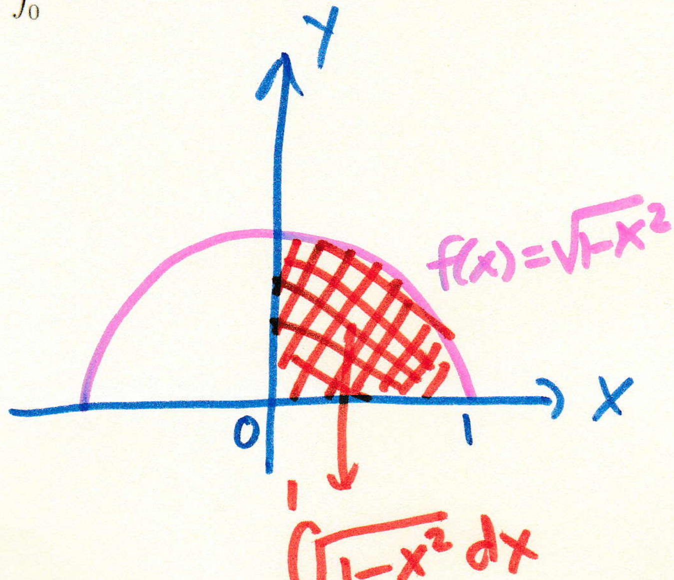
(a) $\int_1^4 2dx$ $\quad f(x) = 2$



พ.ท. $\boxed{\square} = (4-1)\times(2) = 6$

$\therefore \int_1^4 2dx = 6$

(b) $\int_0^1 \sqrt{1-x^2}dx$ $\quad f(x) = \sqrt{1-x^2}$



$\therefore \int_0^1 \sqrt{1-x^2}\,dx = \frac{\pi}{4}$

สรุป

$$\boxed{\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x_k}$$