$f(x) = x^2 - 4 \qquad D_f = \mathbb{R}$

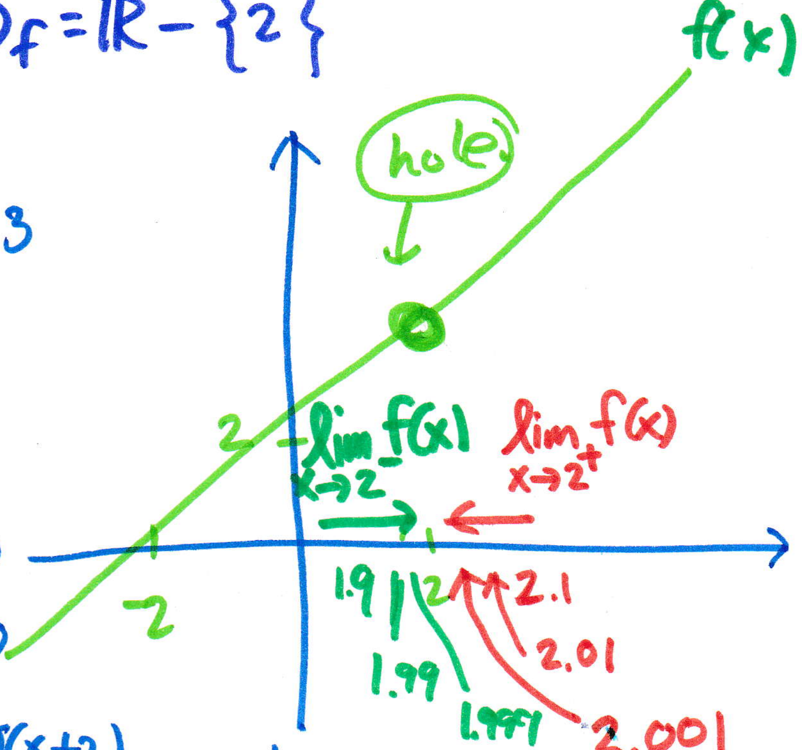
$f(1) = (1)^2 - 4 = -3$

$f(2) = (2)^2 - 4 = 0$

$\boxed{f(x) = \dfrac{x^2-4}{x-2}} \quad D_f = \mathbb{R} - \{2\}$

$f(1) = \dfrac{(1)^2-4}{1-2} = 3$

$f(2) =$ ไม่นิยาม



$n(x) = x^2 - 4 \Rightarrow n(2) = 0$

$d(x) = x - 2 \Rightarrow d(2) = 0$

$f(x) = \dfrac{x^2-4}{(x-2)} = \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} \; ; x \neq 2$

$\downarrow$

$f(x) = (x+2)$ โดยที่ $x \neq 2$

เกิด Hole (รู) ที่ $x = 2$

LIMIT

$\lim_{x\to 2^-} f(x) = 4$

$\lim_{x\to 2^+} f(x) = 4$

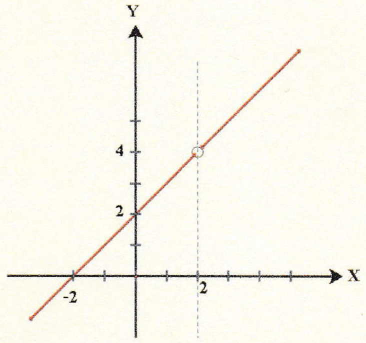
$\boxed{\lim_{x\to 2} f(x) = 4}$ มีค่า (exist)

Chapter 4 Limits and Continuity

4.1 Introduction to Limits

The limit of $f(x)$ at $x = c$

Let $f(x) = \dfrac{x^2 - 4}{x - 2}$. Note that the function f is not defined when $x = 2$. The graph of the function f is



However, we can explore the behavior of $f(x)$ for x near, but not equal to, 2 as the following.

x	1.9	1.99	1.999
$f(x)$	3.9	3.99	3.999

In the above table, when x is close to, but less than, 2, the value $f(x)$ is close to ....4.... .

x	2.1	2.01	2.001
$f(x)$	4.1	4.01	4.001

In the above table, when x is close to, but greater than, 2, the value $f(x)$ is close to ....4.... .

The **limit from the left or the left-handed limit** of $f(x)$ as x approaches c from the left is denoted by

$$\lim_{x \to c^-} f(x).$$

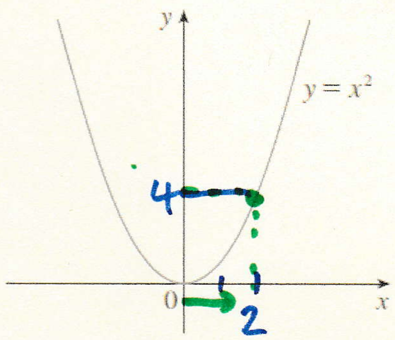
The **limit from the right or the right-handed limit** of $f(x)$ as x approaches c from the right is denoted by

$$\lim_{x \to c^+} f(x).$$

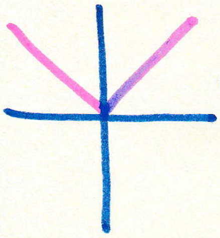
To find one-sided limits when x is close to c, we only consider the value $f(x)$ when x is near c without considering the value of $f(x)$ at c. In fact, c **doesn't need** to be in the domain of f!

> **Limit**
>
> If $\lim_{x \to c^-} f(x) = L = \lim_{x \to c^+} f(x)$, we call L **the limit of a function** f **as** x **approaches** c , denoted by
>
> $$\lim_{x \to c} f(x) = L,$$
>
> where L is a real number. Otherwise, we say that the limit does not exist.

Example 1. Let $f(x) = x^2$. Find $\lim_{x \to 2^-} f(x) =$ ....4.... and $\lim_{x \to 2^+} f(x) =$ ....4....



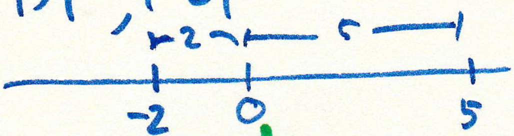
Does $\lim_{x \to 2} f(x)$ exist? Explain. Yes เพราะ $\lim_{x \to 2^-} f(x) = 4 = \lim_{x \to 2^+} f(x)$



$|x| = \begin{cases} x & ; x \geq 0 \\ -(x) & ; x < 0 \end{cases}$

$|-2| = -(-2) = 2$

$|5|$, $|-2|$



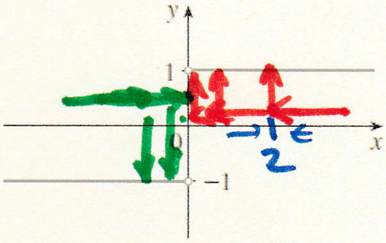
Example 2. Let $f(x) = \dfrac{|x|}{x}$. Find $\lim\limits_{x\to 0^-} f(x) =$ ....-1........ and $\lim\limits_{x\to 0^+} f(x) =$ ....1........

$D_f = \mathbb{R} - \{0\}$

$f(x) = \begin{cases} \dfrac{x}{x} & ; x > 0 \\ -\dfrac{x}{x} & ; x < 0 \end{cases}$

$\downarrow$

$f(x) = \begin{cases} 1 & ; x > 0 \\ -1 & ; x < 0 \end{cases}$



$\lim\limits_{x\to 2} f(x)$

$\hookrightarrow \lim\limits_{x\to 2^+} f(x) = 1$

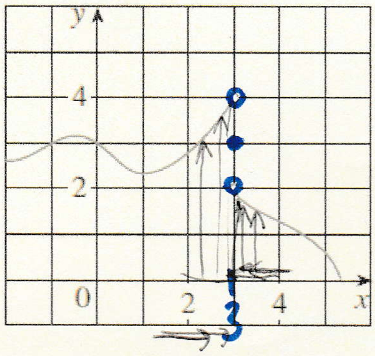
$\lim\limits_{x\to 2^-} f(x) = 1$

$\therefore \lim\limits_{x\to 2} f(x) = 1$

Does $\lim\limits_{x\to 0} f(x)$ exist? Explain.

$\lim\limits_{x\to 0} f(x)$ does not exist because $\lim\limits_{x\to 0^-} f(x) = -1 \neq 1 = \lim\limits_{x\to 0^+} f(x)$

Example 3. Given the graph of the function $f(x)$, find each limit.



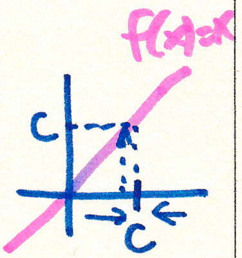
1. $\lim\limits_{x\to 3^-} f(x) =$ 4
2. $\lim\limits_{x\to 3^+} f(x) =$ 2
3. $\lim\limits_{x\to 3} f(x) =$ does not exist ไม่มีค่าลิมิต
4. $f(3) =$ 3

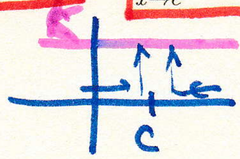
Does $\lim\limits_{x\to 3} f(x)$ exist? Explain.

Limits: An Algebraic Approach

Properties of Limits

Let f and g be two functions, and assume that $\boxed{\lim\limits_{x \to c} f(x) = L}$ and $\boxed{\lim\limits_{x \to c} g(x) = M}$, where L, M and c are real numbers. Then





1. $\lim\limits_{x \to c} \overset{f(x)}{k} = k$ for any constant k
2. $\lim\limits_{x \to c} \overset{f(x)}{x} = c$
3. $\lim\limits_{x \to c} [f(x) \pm g(x)] = \lim\limits_{x \to c} f(x) \pm \lim\limits_{x \to c} g(x) = L \pm M$
4. $\lim\limits_{x \to c} [k \cdot f(x)] = k \cdot \lim\limits_{x \to c} f(x) = k \cdot L$ for any constant k
5. $\lim\limits_{x \to c} [f(x) \cdot g(x)] = \lim\limits_{x \to c} f(x) \cdot \lim\limits_{x \to c} g(x) = L \cdot M$
6. $\lim\limits_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim\limits_{x \to c} f(x)}{\lim\limits_{x \to c} g(x)} = \dfrac{L}{M}$ if $M \neq 0$
7. $\lim\limits_{x \to c} \sqrt[n]{f(x)} = \sqrt[n]{\lim\limits_{x \to c} f(x)} = \sqrt[n]{L}$ (If n is even, $L \geq 0$.)
8. $\lim\limits_{x \to c} [f(x)]^n = [\lim\limits_{x \to c} f(x)]^n$, where n is a positive integer.

Each property above is also valid if $x \to c$ is replaced everywhere by either $x \to c^-$ or $x \to c^+$.

Example 4. Find each limit.

1. $\lim\limits_{x \to 2} (3x + 5)$

PP.3

$$\overset{\text{PP.3}}{=} \lim_{x \to 2} 3x + \lim_{x \to 2} 5$$
$$= 3 \lim_{x \to 2} x + \lim_{x \to 2} 5$$
$$= 3(2) + 5 = 11$$

2. $\lim\limits_{x \to 3} \sqrt[3]{3x - 1}$

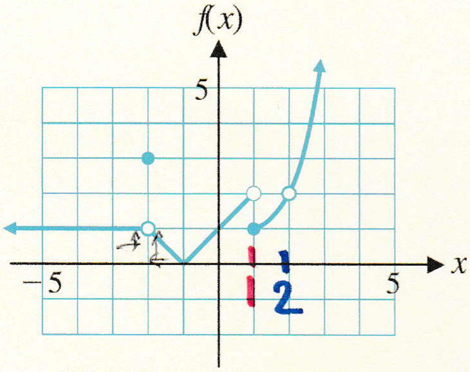
PP.7

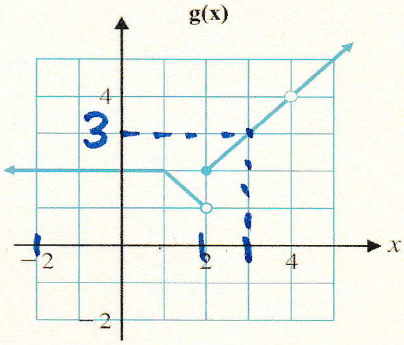
$$= \sqrt[3]{\lim_{x \to 3} (3x-1)} = \sqrt[3]{\lim_{x \to 3} 3x - \lim_{x \to 3} 1}$$
$$= \sqrt[3]{3 \lim_{x \to 3} x - \lim_{x \to 3} 1}$$
$$= \sqrt[3]{9-1} = \sqrt[3]{8} = 2 \ \#$$

3. $\lim\limits_{x \to -1} \dfrac{x}{x + 2}$

$$= \frac{\lim\limits_{x \to -1} x}{\lim\limits_{x \to -1} x+2} = \frac{-1}{1} = -1 \ \#$$

Example 5. Given the graphs of functions $f(x)$ and $g(x)$ as the following figures. Find each limit(if exist). If the limit does not exist, explain.





1. $\lim\limits_{x\to3}[\frac{g(x)}{4}]$ $= \frac{1}{4}\lim\limits_{x\to3} g(x) = \frac{3}{4}$ #

2. $\lim\limits_{x\to2}[x^3 f(x)]$

$\because \lim\limits_{x\to2} f(x) = 2$

$\therefore \lim\limits_{x\to2}[x^3 f(x)] = (\lim\limits_{x\to2} x^3)(\lim\limits_{x\to2} f(x))$

$= 8 \cdot 2 = 16$ #

3. $\lim\limits_{x\to1}[f(x)]$

$\left.\begin{array}{l}\lim\limits_{x\to1^-} f(x) = 2\\ \lim\limits_{x\to1^+} f(x) = 1\end{array}\right\}$ $\therefore \lim\limits_{x\to1} f(x)$ does not exist

because $\lim\limits_{x\to1^-} f(x) = 2 \neq 1 = \lim\limits_{x\to1^+} f(x)$

4. $\lim\limits_{x\to-2}[f(x)+g(x)]$

$\lim\limits_{x\to-2} g(x) = 2$

$\lim\limits_{x\to-2} f(x) = 1$

$\therefore \lim\limits_{x\to-2}[f(x)+g(x)] = \lim\limits_{x\to-2} f(x) + \lim\limits_{x\to-2} g(x) = 2+1$

$= 3$ #

5. $\lim\limits_{x\to2}[f(x)+g(x)]$

$\left.\begin{array}{l}\lim\limits_{x\to2^+} g(x) = 2\\ \lim\limits_{x\to2^-} g(x) = 1\end{array}\right\}$ $\lim\limits_{x\to2} g(x)$ ไม่มี

$\therefore \lim\limits_{x\to2}[f(x)+g(x)]$ ไม่มี

6. $\lim\limits_{x\to1}[f(x)+g(x)]$

ตาม 3. $\lim\limits_{x\to1} f(x)$ ไม่มี

$\therefore \lim\limits_{x\to1}[f(x)+g(x)]$ ไม่มี

Limits of Polynomial Functions

Let c be a real number and $f(x)$ a polynomial function defined by

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

where $a_0, a_1, \ldots, a_n$ are real numbers. Then

$$\lim_{x\to c} f(x) = f(c).$$

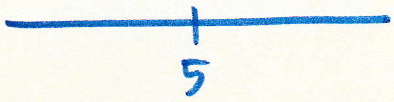
Example 6. Find each limit.

1. $\lim_{x \to -5} 10$ $= 10$

2. $\lim_{x \to 4} (x^2 - 4x + 3) = (4)^2 - 4(4) + 3$
$= 3$ #

Example 7. Find each limit.

1. $f(x) = \begin{cases} \boxed{2x + 3} & \text{if } x < 5 \\ \boxed{-x + 12} & \text{if } x > 5 \end{cases}$



piecewise-defined function

(a) $\lim_{x \to 5^-} f(x) = \lim_{x \to 5^-} (2x+3)$
$= 2(5) + 3 = 13$

(b) $\lim_{x \to 5^+} f(x) = \lim_{x \to 5^+} (-x+12)$
$= -(5) + 12 = 7$

(c) $\lim_{x \to 5} f(x)$ does not exist
because $\lim_{x \to 5^-} f(x) = 13 \neq 7 = \lim_{x \to 5^+} f(x)$

(d) $f(5)$ does not exist.

2. $f(x) = \begin{cases} x & \text{if } x < 0 \\ x^2 & \text{if } 0 < x \leq 2 \\ 8 - x & \text{if } x > 2 \end{cases}$

(a) $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x = 0$

(b) $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^2 = 0$

(c) $\lim_{x \to 0} f(x) = 0$ ส่ก/ครีง

(d) $f(0)$ does not exist.

(e) $\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} x^2 = 4$

(f) $\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} 8 - x = 6$

(g) $\lim_{x \to 2} f(x)$ does not exist

(h) $\boxed{f(2)}$ $= (2)^2 = 4$

Limits of Rational Functions

Let c be a real number and $f(x)$ a rational function defined by $f(x) = \dfrac{n(x)}{d(x)}$, where $n(x)$ and $d(x)$ are polynomial functions.

1. If $d(c) \neq 0$, then
$$\lim_{x\to c} f(x) = f(c) = \frac{n(c)}{d(c)}.$$
$\cancel{\frac{0}{0} = 1}$

2. If $d(c) = 0$ and $n(c) = 0$ $\left(\dfrac{0}{0}\ \textbf{Indeterminate Form}\right)$, we have to simplify $f(x)$ by canceling out any common factors of $n(x)$ and $d(x)$ and then find the limit.

รูปแบบไม่กำหนด

The case when $d(c) = 0$ and $n(c) \neq 0$ will be discussed in Section 4.2.

Example 8. Find each limit.

1. $\displaystyle\lim_{x\to 2} \frac{5x^3+4}{x-3}$ ← $n(x)$, ← $d(x)$

$d(2) \neq 0$; $\displaystyle\lim_{x\to 2} \frac{5x^3+4}{x-3} = \frac{5(8)+4}{2-3}$
$= -44$

2. $\displaystyle\lim_{x\to 2} \frac{x^2-4}{x-2}$ ← $n(x)$, ← $d(x)$

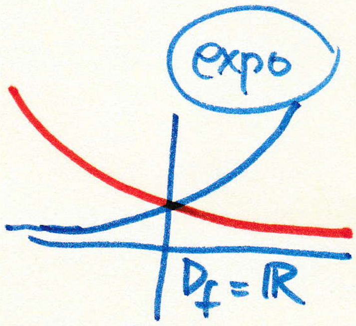
$d(2) = 0$, $n(2) = 0$

$\displaystyle\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$
$= \displaystyle\lim_{x\to 2} (x+2)$
$= 2+2 = 4$

3. $\displaystyle\lim_{x\to -4} \left(\frac{2x+8}{x^2+x-12}\right)$ ← $n(x)$, ← $d(x)$

$d(-4) = 0$, $n(-4) = 0$

$\displaystyle\lim_{x\to -4} \frac{2\cancel{(x+4)}}{\cancel{(x+4)}(x-3)} = \lim_{x\to -4} \frac{2}{x-3}$
$= -\dfrac{2}{7}$



Example 9. Find each limit.

1. $\displaystyle\lim_{x\to 3} e^x$ $= e^3$

2. $\displaystyle\lim_{x\to 3} e^{-x}$ $= \dfrac{1}{e^3}$

$e^{-x} = (e^{-1})^x = \left(\dfrac{1}{e}\right)^x$

3. $\displaystyle\lim_{x\to 5} \ln x$ $= \ln 5$

4. $\displaystyle\lim_{x\to 2} \ln(6x)$ $= \ln 12$