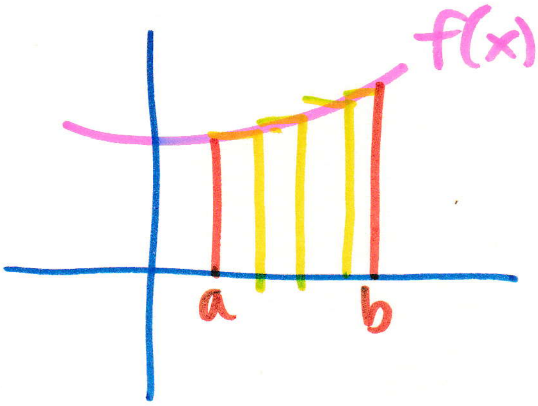
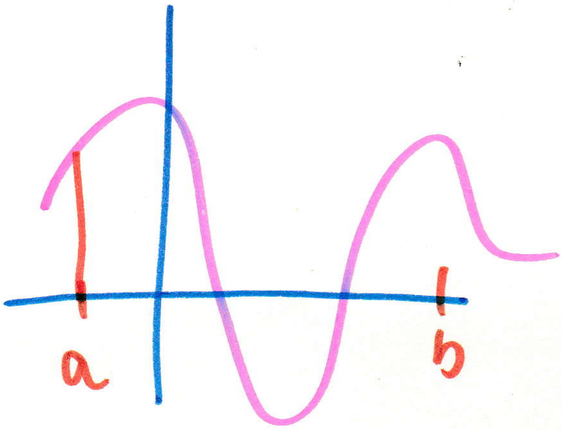


$f(x) \geq 0$

ผลบวกรีมันน์ :

Riemann's Integral



$$\boxed{\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n}} f(x_k^*)\,\Delta x_k$$

$$\|$$

$$A = \int_a^b f(x)\,dx$$

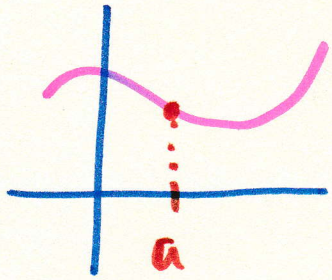
Definite integral

5.3.2 Properties of the Definite Integral

Theorem 5.3

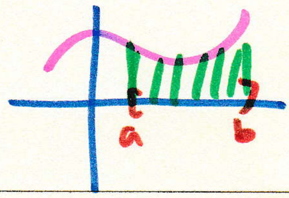
(a) If a is in the domain of f, we define

$$\int_a^a f(x)dx = 0$$



(b) If f is integrable on $[a, b]$, then we define

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$



Example 5.4

(a) $\int_1^1 (\sin 1 - x^2)dx = 0$

(b) $\int_1^0 (\sqrt{1-x^2})dx = -\int_0^1 \sqrt{1-x^2}\,dx = -\frac{\pi}{4}$

Theorem 5.4 If f and g are integrable on $[a, b]$ and if c is a constant, then cf, $f+g$, and fg are integrable on $[a, b]$ and

(a) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$

(b) $\int_a^b f(x) + g(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

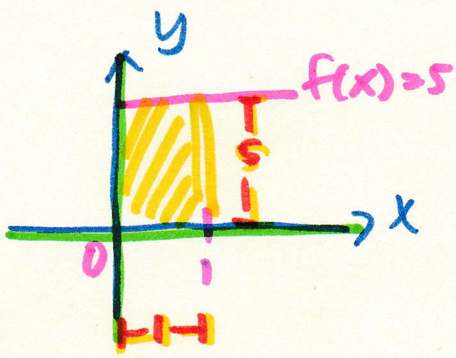
(c) $\int_a^b f(x) - g(x)dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

Example 5.5 Evaluate $\int_0^1 (5 - 3\sqrt{1-x^2})dx$

$$\int_0^1 (5-3\sqrt{1-x^2})dx = \int_0^1 5dx - \int_0^1 3\sqrt{1-x^2}\,dx$$

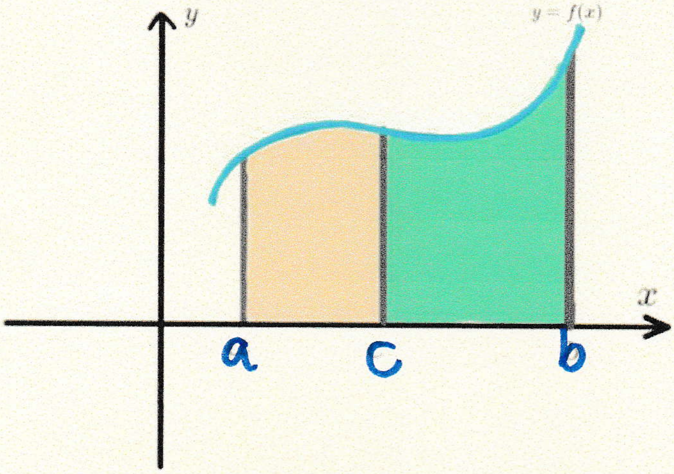
$$= \int_0^1 5dx - 3\int_0^1 \sqrt{1-x^2}\,dx$$

$$= 5 - 3\left(\frac{\pi}{4}\right) \quad \#$$



Theorem 5.5 If f is integrable on a closed interval containing the three points $a, b,$ and c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$



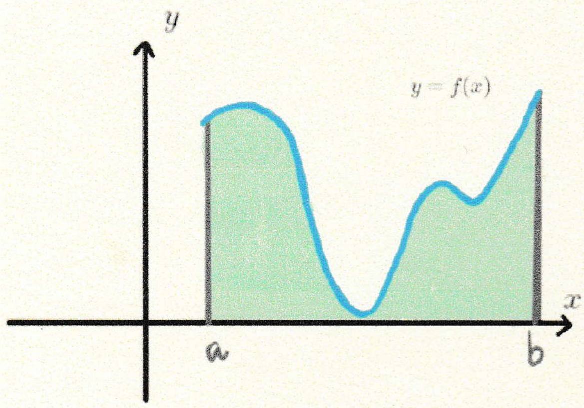
Theorem 5.6

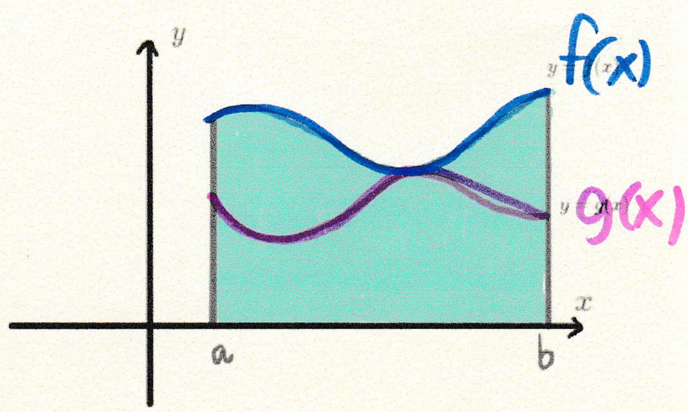
(a) If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx \geq 0$$

(b) If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$





$f(x) = x$

หา $F(x)$ ที่ทำให้ $F'(x) = f(x)$

$F(x) = \frac{x^2}{2} + C$ $\begin{cases} F(x) = \frac{x^2}{2} & F'(x) = \frac{2x}{2} = f(x) \\ F(x) = \frac{x^2}{2} + 5 & F'(x) = x^2 = f(x) \end{cases}$

$f(x) = \cos x$

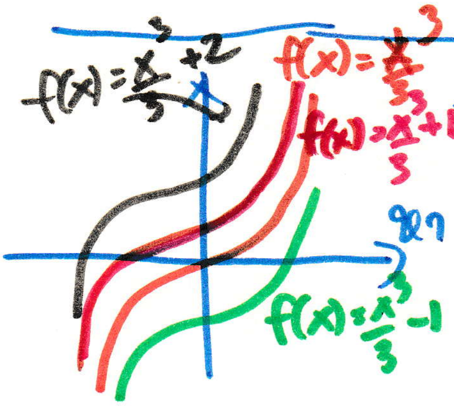
หา $F(x)$ ที่ทำให้ $F'(x) = \cos x$

เลือก $F(x) = \sin x$ $\quad F'(x) = \cos x = f(x)$

$F(x) = \sin x + \frac{\pi}{2}$ $\quad F'(x) = \cos x = f(x)$

$F(x) = \sin x + \frac{1001\pi}{1010}$ $\quad F'(x) = \cos x = f(x)$

$F(x) = \sin x + C$ เมื่อ C เป็นค่าคงที่ (constant)



$f(x) = x^2$

หา $F(x)$ ที่ทำให้ $F'(x) = x^2$

$F(x) = \frac{x^3}{3} + C$

Ex $F(x)$ ผ่านจุด $(\_, \_)$

ปฏิยานุพันธ์

5.3.3 The Antiderivative Method for Finding Areas

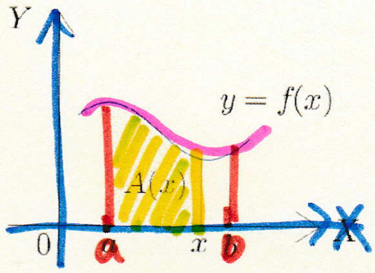
> DEFINITION A function F is called an *antiderivative* of a function f on a given open interval if $F'(x) = f(x)$ for all x in the interval.

For example, $\frac{1}{3}x^3, \frac{1}{3}x^3 + 1, \frac{1}{3}x^3 - 3, \frac{1}{3}x^3 - \sqrt{2}$ are all antiderivatives of $f(x) = \ldots\ldots\ldots$

If f is a nonnegative continuous function on the interval $[a, b]$, and if $A(x)$ denotes the area under the graph of f over the interval $[a, x]$, where x is any point in the interval $[a, b]$ (Figure 5.1.4), then

$$A'(x) = f(x) \tag{5.2}$$

The following example confirms Formula (5.2) in some cases where a formula for $A(x)$ can be found using elementary geometry.



A(x) คือฟังก์ชันของ พ.ท.
จาก a ถึง x
เมื่อ $x \in [a,b]$

Example 5.6 For each of the functions f, find the area $A(x)$ between the graph of f and the interval $[a, x]$, and find the derivative $A'(x)$ of this area function.

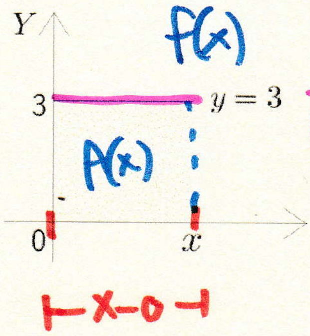
Solution

(a) $f(x) = 3; \quad a = 0$

ความกว้าง ความสูง

$A(x) = (x-0) \cdot 3$

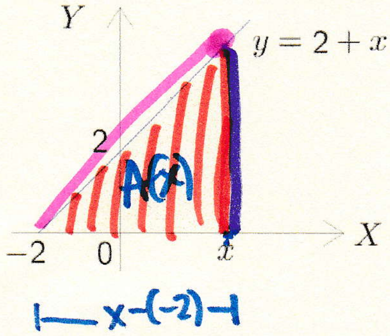
$A(x) = 3x$



} ความสูง (f(x)

$A'(x) = 3$

(b) $f(x) = 2 + x; \quad a = -2$



$$A(x) = \frac{1}{2} \times \overset{\text{ฐาน}}{(x+2)} \times \overset{\text{สูง}}{(2+x)}$$

$$= \frac{1}{2}(x+2)^2$$

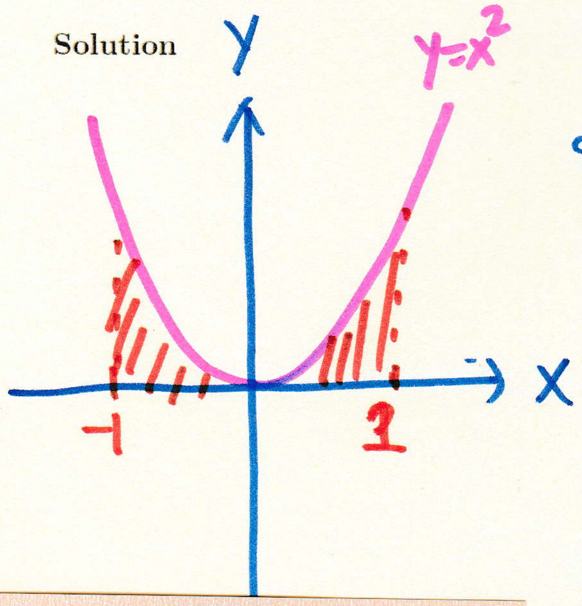
$$A'(x) = x+2$$

$$\Downarrow$$

$$f(x) = 2+x = x+2$$

Example 5.7 Use the antiderivative method to find the area under the graph of $y = x^2$ over the interval $[-1, 1]$.

Solution



ความหมาย: $A'(x) = x^2$

หาฟังก์ชัน $A(x)$ ซึ่ง $A'(x) = x^2$

$$\boxed{A(x) = \frac{x^3}{3} + C}$$

ฟังก์ชัน พ.ท. ของเรา ดูจาก ช่วง $[-1, x]$

หาค่า C ยังไงดี?

กรณีไหนที่ $A(x) = 0$?

พิจารณา $0 = A(-1) = \frac{(-1)^3}{3} + C$

$$0 = -\frac{1}{3} + C$$

$$\therefore \boxed{C = \frac{1}{3}}$$

$$\boxed{A(x) = \frac{x^3}{3} + \frac{1}{3}}$$

← เป็นฟังก์ชันที่ใช้หา พ.ท. ใต้ $y = x^2$ จาก ~~ต~~ -1 ถึง x

เพิ่มเติมจากในคาบ.

• $A(x) = \frac{x^3}{3} + \frac{1}{3}$ คือฟังก์ชันพื้นที่ใต้กราฟ $y = x^2$ จาก -1 ถึง x
ดังนั้นหากต้องการหาพื้นที่บนช่วง $[-1, 1]$
เราจะต้องพิจารณา $A(1) = \frac{(1)^3}{3} + \frac{1}{3} = \frac{2}{3}$
ซึ่งจะเห็นว่า สอดคล้องกับ ค่าลิมิตของ Riemann Sum ในคาบที่แล้ว นั่นคือ

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} f\left(-1 + k\left(\frac{1}{n}\right)\right)\frac{1}{n} = \lim_{n\to\infty} \sum_{k=0}^{n-1} \left(-1 + \frac{k}{n}\right)^2 \left(\frac{1}{n}\right) = \frac{2}{3}$$

นี่จึงเป็นอีก Approach หนึ่งของการหาพื้นที่ใต้ $y = x^2$ บน $[-1, 1]$

5.4 The Indefinite Integral

> **Theorem 5.7** If $F(x)$ is any antiderivative of $f(x)$ on an open interval, then for any constant C the function $F(x) + C$ is also an antiderivative on that interval. Moreover, each antiderivative of $f(x)$ on the interval can be expressed in the form $F(x) + C$ by choosing the constant C appropriately.

หา $F(x)$ ที่ดิฟแล้วได้ $f(x)$ โดยก็ต้องบวก C ด้วย

5.4.1 The Indefinite Integral

The process of finding antiderivatives is called **antidifferentiation** or **integration**. Thus, if

$$\frac{d}{dx}[F(x)] = f(x) \tag{5.3}$$

then **integrating** (or **antidifferentiating**) the function $f(x)$ produces an antiderivative of the form $F(x) + C$. To emphasize this process, Equation (5.3) is recast using integral notation,

$$\int f(x)dx = F(x) + C \tag{5.4}$$

Ex $F(x) = \sqrt{x^2-1}$

$F'(x) = \dfrac{2x}{2\sqrt{x^2-1}} = \dfrac{x}{\sqrt{x^2-1}}$

หา $\displaystyle\int \frac{x}{\sqrt{x^2-1}}\,dx$ = หา $F(x)$ ซึ่ง $F'(x) = \underbrace{\dfrac{x}{\sqrt{x^2-1}}}_{f(x)}$

$\therefore \displaystyle\int \frac{x}{\sqrt{x^2-1}}\,dx = \sqrt{x^2-1} + C$

หา $F(x)$ ซึ่ง $F'(x) = f(x)$

$\Updownarrow$

$\boxed{\int f(x)dx = F(x) + C}$

5.4.2 Integration Formulas

Differentiation Formula	Integration Formula
1. $\frac{d}{dx}(C) = 0$	1. $\int 0dx = C$
2. $\frac{d}{dx}[kx] = k$	2. $\int kdx = kx + C$
3. $\frac{d}{dx}[kf(x)] = kf'(x)$	3. $\int [kf(x)]dx = k\int f(x)dx$
4. $\frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$	4. $\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx$
5. $\frac{d}{dx}[x^n] = nx^{n-1}$	5. $\int x^n dx = \frac{x^{n+1}}{n+1} + C,\ n \neq -1$
6. $\frac{d}{dx}[\ln\|x\|] = \frac{1}{x}$	6. $\int \frac{1}{x}dx = \ln\|x\| + C$
7. $\frac{d}{dx}[e^x] = e^x$	7. $\int e^x dx = e^x + C$
8. $\frac{d}{dx}[a^x] = a^x \ln a,\ a > 0$ and $a \neq 1$	8. $\int a^x dx = \frac{a^x}{\ln a} + C,\ a > 0$ and $a \neq 1$
9. $\frac{d}{dx}[\sin x] = \cos x$	9. $\int \cos x dx = \sin x + C$
10. $\frac{d}{dx}[\cos x] = -\sin x$	10. $\int \sin x dx = -\cos x + C$
11. $\frac{d}{dx}[\tan x] = \sec^2 x$	11. $\int \sec^2 x dx = \tan x + C$
12. $\frac{d}{dx}[\cot x] = -\mathrm{cosec}^2 x$	12. $\int \mathrm{cosec}^2 x dx = -\cot x + C$
13. $\frac{d}{dx}[\sec x] = \sec x \tan x$	13. $\int \sec x \tan x dx = \sec x + C$
14. $\frac{d}{dx}[\mathrm{cosec}\, x] = -\mathrm{cosec}\, x \cot x$	14. $\int \mathrm{cosec}\, x \cot x dx = -\mathrm{cosec}\, x + C$
15. $\frac{d}{dx}[\arcsin x] = \frac{1}{\sqrt{1-x^2}}$	15. $\int \frac{1}{\sqrt{1-x^2}}dx = \arcsin x + C$ → $\sin^{-1}x$
16. $\frac{d}{dx}[\arctan x] = \frac{1}{1+x^2}$	16. $\int \frac{1}{1+x^2}dx = \arctan x + C$ → $\tan^{-1}x$
17. $\frac{d}{dx}[\ln\|\sec x\|] = \tan x$	$\boxed{17.}$ $\int \tan x dx = \ln\|\sec x\| + C$
18. $\frac{d}{dx}[\ln\|\sin x\|] = \cot x$	$\boxed{18.}$ $\int \cot x dx = \ln\|\sin x\| + C$

Example 5.8

(a) $\int (x^6 - 7x + 4)dx = \int x^6 dx - \int 7x\,dx + \int 4\,dx$

$$= \left(\frac{x^{6+1}}{6+1} + C_1\right) - 7\left(\frac{x^2}{2} + C_2\right) + (4x + C_3)$$

$$= \frac{x^7}{7} - \frac{7x^2}{2} + 4x + C_1 - C_2 + C_3 \quad + C$$

$$= \frac{x^7}{7} - \frac{7}{2}x^2 + 4x + C \quad (\therefore C = C_1 - C_2 + C_3)$$

(b) $\int \frac{x^5 + 2x^3 - 1}{x^4}dx = \int (x + \frac{2}{x} - \frac{1}{x^4})dx$

$$\cancel{\int \frac{2}{x}dx = \int 2x^{-1}dx = \frac{2x^{-1+1}}{-1+1} + C}$$

ใช้สูตร (5)

$$= \frac{x^2}{2} + 2\ln|x| - \frac{x^{-4+1}}{(-4+1)} + C$$

$$= \frac{x^2}{2} + 2\ln|x| + \frac{x^{-3}}{3} + C$$

$$= \frac{x^2}{2} + 2\ln|x| + \frac{1}{3x^3} + C \quad \#$$

(c) $\int (\sqrt{x} + \frac{1}{\sqrt[3]{x}})dx = \int (x^{\frac{1}{2}} + \frac{1}{x^{\frac{1}{3}}})dx$

$$= \frac{x^{1+\frac{1}{2}}}{1+\frac{1}{2}} + \frac{x^{-\frac{1}{3}+1}}{-\frac{1}{3}+1} + C$$

$$= \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{x^{+\frac{2}{3}}}{+\frac{2}{3}} + C = \frac{2}{3}x^{\frac{3}{2}} + \frac{3}{2}x^{\frac{2}{3}} + C \quad \#$$

$$\cancel{= \frac{2x^{\frac{3}{2}}}{3} + \frac{3}{2}\frac{1}{x^{\frac{2}{3}}} + C \quad \#}$$

(d) $\int(e^x + 2^x)dx =$ $e^x + \frac{2^x}{\ln 2} + C$ #

7 8

(e) $\int(4\sin x + 2\cos x)dx =$ $-4\cos x + 2\sin x + C$ #

(f) $\int(\frac{3}{\sqrt{1-x^2}} - \frac{2}{1+x^2})dx =$ $3\arcsin x - 2\arctan x + C$ #

5.5 Integration by Substitution

Theorem 5.8 Let u be a function of x and f be a function of u. Then

$$\int [f(u)]du = \int [f(u(x))u'(x)]dx.$$

Example 5.9 Evaluate $\int (2x+1)^{100}dx$

สูตร 5 $\int x^n dx = \frac{x^{n+1}}{n+1} + C$

วิธีทำ พยายามจัดรูปให้เป็น $\underbrace{\int x^n dx}_{\int u^n du}$

$\therefore$ เลือก $u(x)$ ที่เหมาะสม เพื่อทำการเปลี่ยนตัวแปรในการอินทิเกรต

เป้าหมาย $\int u^{100} du$

differential

เลือก $u = 2x+1 \Rightarrow \boxed{du = 2dx}$

$\therefore \int (2x+1)^{100}dx = \frac{1}{2}\int 2u^{100}dx$ du $= \frac{1}{2}\int u^{100}du = \frac{1}{2}\frac{u^{101}}{101} + C$

$= \frac{1}{2}\int 2(2x+1)^{100}dx$

$= \frac{1}{2}\frac{(2x+1)^{101}}{101} + C$ #

Example 5.10 Evaluate $\int \frac{x^2}{(3x^3-2)^9}dx$

อีกวิธี

$u = 2x+1 \qquad du = 2dx \Rightarrow dx = \frac{1}{2}du$

$$\therefore \int (2x+1)^{100}\,dx = \int u^{100}\left(\frac{1}{2}\right)du = \frac{1}{2}\int u^{100}\,du$$

$$= \frac{1}{2}\cdot\frac{u^{101}}{101} + C$$

$$= \frac{1}{2}\,\frac{(2x+1)^{101}}{101} + C \quad \#$$