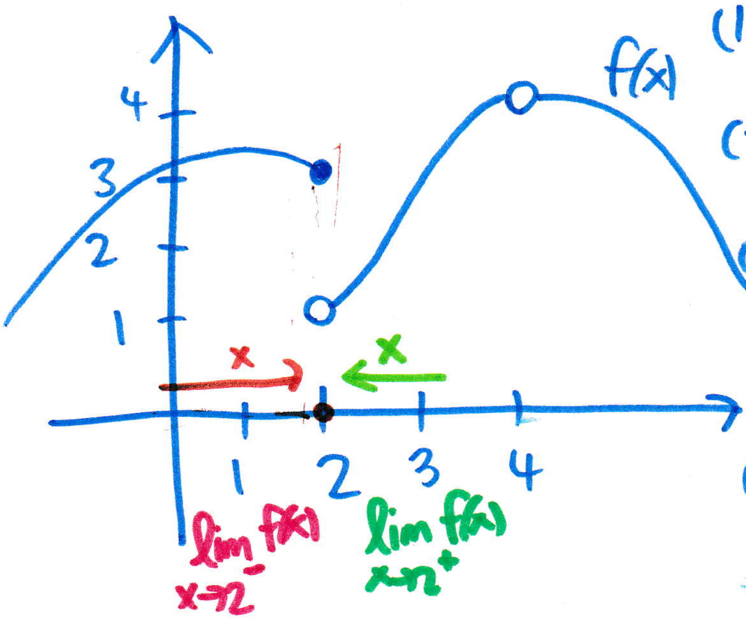
ทบทวน 1



(1) $\lim_{x\to 2^-} f(x) = 3$

(2) $\lim_{x\to 2^+} f(x) = 1$

(3) $\lim_{x\to 2} f(x) =$ ไม่มีค่า/ครับ

(4) $f(2) = 3$

(5) $f(4) =$ ไม่นิยาม

(6) $\lim_{x\to 4} f(x) = 4$

2 จงหาลิมิต

$$\lim_{x\to 1} \frac{x^2-2x+1}{x^2-1} = \lim_{x\to 1} \frac{\cancel{(x-1)}(x-1)}{\cancel{(x-1)}(x+1)} = \lim_{x\to 1} \frac{x-1}{x+1} = 0$$

$n(1)=0$

$d(1)=0$

3 ให้ $\lim_{x\to 0} f(x) = 2$, $\lim_{x\to 0} g(x) = -2$

จงหา (1) $\lim_{x\to 0} \sqrt{\dfrac{g(x)+1}{f(x)\cdot g(x)}} = \sqrt{\dfrac{(-2)+1}{2\cdot(-2)}} = \sqrt{\dfrac{1}{4}} = \dfrac{1}{2}$

(2) $\lim_{x\to 0} \left\{ [f(x)]^3 + \frac{1}{2} g(x) \right\} = \lim_{x\to 0} [f(x)]^3 + \frac{1}{2} \lim_{x\to 0} g(x)$

$= \left[\lim_{x\to 0} f(x)\right]^3 + \frac{1}{2} \lim_{x\to 0} g(x)$

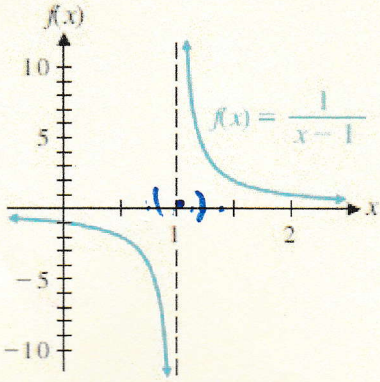
$= [2]^3 + \frac{1}{2}((-2))$

$= 8 - 1 = 7$ #

4.2 Infinite Limits and Limits at Infinity

Infinite Limits ลิมิตอนันต์

Let $f(x) = \dfrac{1}{x-1}$. Note that there does not exist a real number L that the values of $f(x)$ approach as x approaches 1. The graph of the function f is



$D_f = \mathbb{R} - \{1\}$

However, we can explore the behavior of $f(x)$ for x near, but not equal to, 1 as the following.

$f(0.9) = \frac{1}{0.9-1} = -\frac{1}{0.1} = -10$ $f(0.99) = \frac{1}{0.99-1} = -\frac{1}{0.01} = -100$

x	0.9	0.99	0.999
$f(x)$	-10	-100	-1000

In the above table, when x approaches 1 from the left, the value $f(x)$ decreases without bound. We write

$$\lim_{x\to 1^-} f(x) = -\infty$$

$f(1.1) = \frac{1}{1.1-1} = \frac{1}{0.1} = 10$

x	1.1	1.01	1.001
$f(x)$	10	100	1,000

In the above table, when x approaches 1 from the right, the value $f(x)$ increases without bound. We write

$$\lim_{x\to 1^+} f(x) = +\infty$$

Note that the symbol ∞ and the symbol $-\infty$ are used to describe the behavior of $f(x)$ that increases and decreases without bound, respectively.

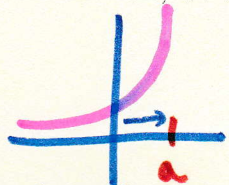
Since both ∞ and $-\infty$ are NOT real numbers, both one-sided limits do not exist and call these situations infinite limits. Also, we say that

$$\lim_{x\to 1} \frac{1}{x-1} \text{ does not exist.}$$

Infinite Limit

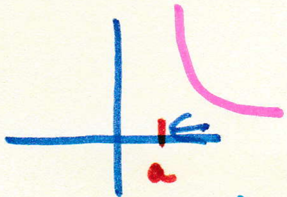
1. If $f(x)$ increases without bound as x approaches a from the left, we write

$$\lim_{x\to a^-} f(x) = \infty$$



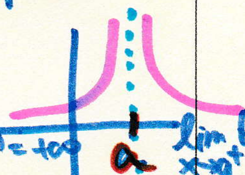
2. If $f(x)$ increases without bound as x approaches a from the right, we write

$$\lim_{x\to a^+} f(x) = \infty$$



3. We write $\lim_{x\to a} f(x) = \infty$ if

$$\lim_{x\to a^-} f(x) = \infty \text{ and } \lim_{x\to a^+} f(x) = \infty$$



$\lim_{x\to a^-} f(x) = +\infty$ $\lim_{x\to a^+} f(x) = +\infty$

$\therefore \lim_{x\to a} f(x) = +\infty$

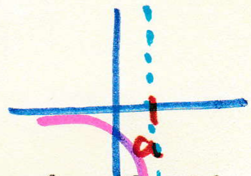
$\lim_{x\to a^-} f(x) = +\infty$

$\lim_{x\to a^+} f(x) = -\infty$

$\therefore \lim_{x\to a} f(x)$ does not exist.

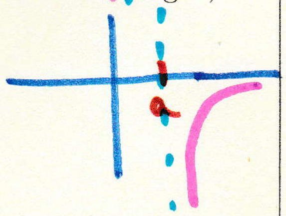
4. If $f(x)$ decreases without bound as x approaches a from the left, we write

$$\lim_{x\to a^-} f(x) = -\infty$$



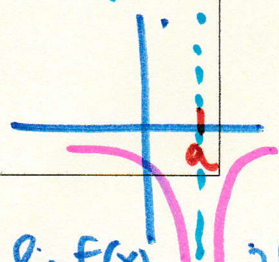
5. If $f(x)$ decreases without bound as x approaches a from the right, we write

$$\lim_{x\to a^+} f(x) = -\infty$$



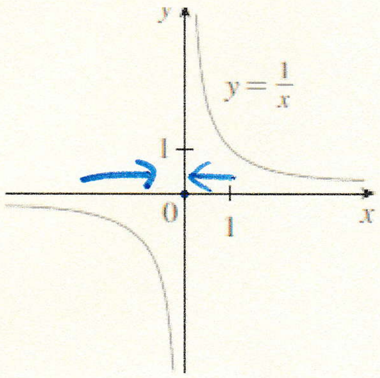
6. We write $\lim_{x\to a} f(x) = -\infty$ if

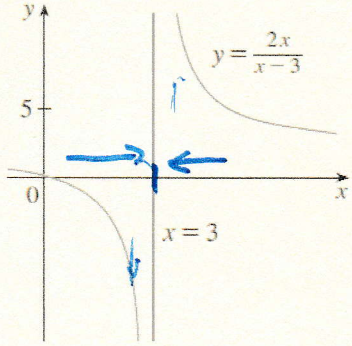
$$\lim_{x\to a^-} f(x) = -\infty \text{ and } \lim_{x\to a^+} f(x) = -\infty$$



$\left.\begin{array}{l}\lim_{x\to a^-} f(x) = -\infty \\ \lim_{x\to a^+} f(x) = -\infty\end{array}\right\} \lim_{x\to a} f(x) = -\infty$

Example 10. Find each limit.





1. $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

3. $\lim_{x \to 0} \frac{1}{x} =$ does not exist

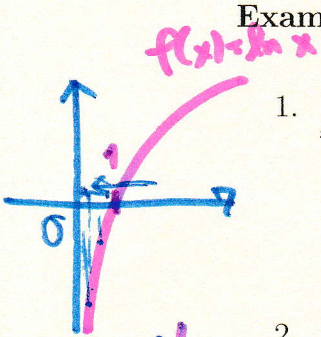
5. $\lim_{x \to 3^+} \frac{2x}{x-3} = +\infty$

2. $\lim_{x \to 0^+} \frac{1}{x} = +\infty$

4. $\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$

6. $\lim_{x \to 3} \frac{2x}{x-3} =$ does not exist.

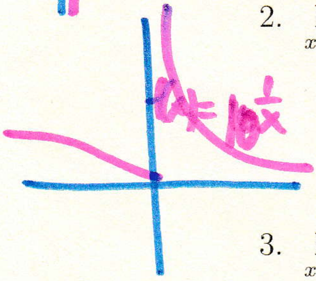
Example 11. Find each limit.



1. $\lim_{x \to 0^+} \ln x = -\infty$

4. $\lim_{x \to 0^+} e^{\frac{1}{x}} =$ $x \to 0^+$, $\frac{1}{x} \to +\infty$

$\therefore \lim_{x \to 0^+} e^{\frac{1}{x}} = +\infty$



2. $\lim_{x \to 0^+} 10^{\frac{1}{x}} =$ $x \to 0^+$, $\frac{1}{x} \to +\infty$

$+\infty$

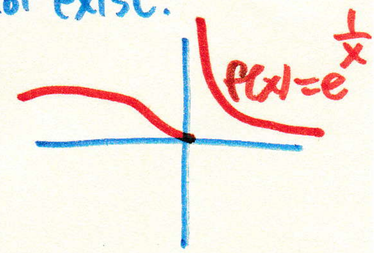
5. $\lim_{x \to 0^-} e^{\frac{1}{x}} =$ $x \to 0^-$, $\frac{1}{x} \to -\infty$

$\therefore \lim_{x \to 0^-} e^{\frac{1}{x}} = 0$

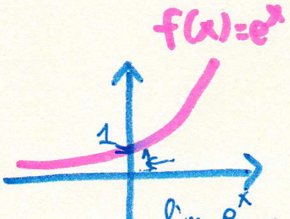
3. $\lim_{x \to 0^-} 5^{\frac{1}{3x}} =$ $x \to 0^-$, $\frac{1}{3x} \to -\infty$

$\therefore \lim_{x \to 0^-} 5^{\frac{1}{3x}} = 0$

6. $\lim_{x \to 0} e^{\frac{1}{x}} =$ does not exist.



$f(x) = e^x$



$\lim_{x \to 0^+} e^x = 1$

$\lim_{x \to 0^-} e^x = \underline{1}$

Infinite Limits and Vertical Asymptotes

The vertical line $x = a$ is a vertical asymptote for the graph of $y = f(x)$ if

$$\lim_{x\to a^-} f(x) = +\infty/-\infty \text{ or } \lim_{x\to a^+} f(x) = +\infty/-\infty$$

Locating Vertical Asymptotes of Rational Functions

If $f(x) = \dfrac{n(x)}{d(x)}$ is a rational function, $d(c) = 0$ and $n(c) \neq 0$, then the line $x = c$ is a vertical asymptote of the graph of f.

Finding the limit of the rational function $f(x) = \dfrac{n(x)}{d(x)}$ as $x \to c$

If $d(c) = 0$ and $n(c) \neq 0$, then $\lim\limits_{x\to c} f(x)$ does not exist or $\lim\limits_{x\to c} f(x) = +\infty/-\infty$.

Example 12. Find each limit.

1. $\lim\limits_{x\to 3^-} \dfrac{-2}{(x-3)^2} = \boxed{-\infty}$

$x \to 3^-$, $\dfrac{1}{x-3} \to -\infty$

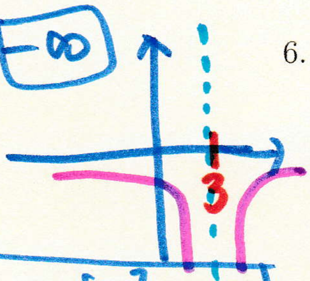
$\left(\dfrac{1}{x-3}\right)^2 \to +\infty$

2. $\lim\limits_{x\to 3^+} \dfrac{-2}{(x-3)^2} = \boxed{-\infty}$

$x \to 3^+$, $\dfrac{1}{x-3} \to +\infty$

$\left(\dfrac{1}{(x-3)}\right)^2 \to +\infty$

3. $\lim\limits_{x\to 3} \dfrac{-2}{(x-3)^2} = \boxed{-\infty}$



$f(x) = -\dfrac{2}{(x-3)^2}$ $D_f = \mathbb{R} - \{3\}$

$x = 3$ เป็น vertical asymptote

4. $\lim\limits_{x\to 4^-} \dfrac{2-x}{(x-4)(x+2)} = \boxed{+\infty}$ (-2)

$x \to 4^-$, $\dfrac{1}{x-4} \to -\infty$

$x \to 4^-$, $\dfrac{1}{x+2} \to \dfrac{1}{2}$

5. $\lim\limits_{x\to 4^+} \dfrac{2-x}{(x-4)(x+2)} = \boxed{-\infty}$ (-2)

$x \to 4^+$, $\dfrac{1}{x-4} \to +\infty$

$x \to 4^+$, $\dfrac{1}{x+2} \to \dfrac{1}{6}$

6. $\lim\limits_{x\to 4} \dfrac{2-x}{(x-4)(x+2)} =$ does not exist.

(4-6) $f(x) = \dfrac{2-x}{(x-4)(x+2)}$

$D_f = \mathbb{R} - \{-2, 4\}$

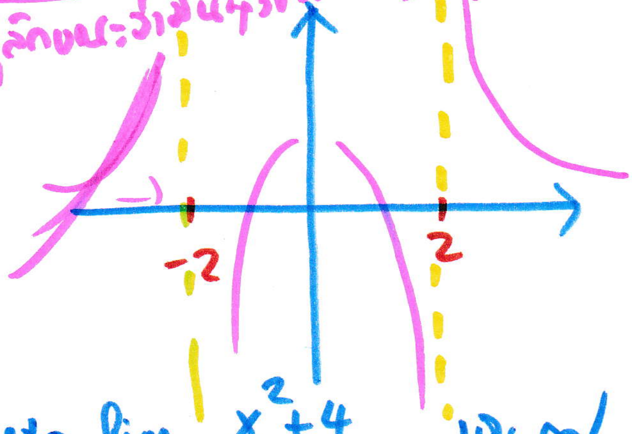
$\therefore x = -2, 4$ เป็น vertical asymptote.

$\lim\limits_{x\to -2^-} \dfrac{2-x}{(x-4)(x+2)} = -\infty$

$\lim\limits_{x\to -2^+} \dfrac{2-x}{(x-4)(x+2)} = +\infty$

$\lim\limits_{x\to -2}$ ~~~ does not exist.

Not to scale
(ดูลักษณะว่าเส้นพุ่งขึ้น/ลง)



$$f(x) = \frac{x^2+4}{x^2-4}$$

$$f(x) = \frac{x^2+4}{(x-2)(x+2)}$$

ว่า $x = 2, -2$ เป็น V.A.

$\boxed{\text{Check}}$

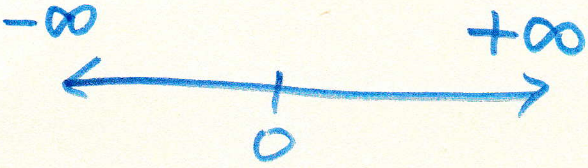
เช็คว่า $x=2$ เป็น V.A.?

$\lim\limits_{x\to 2^+} \frac{x^2+4}{(x-2)(x+2)} = +\infty$ และ $\lim\limits_{x\to 2^-} \frac{x^2+4}{(x-2)(x+2)} = -\infty$ เป็น $\infty/-\infty$

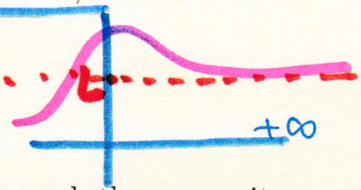
เช็คว่า $x=-2$ เป็น V.A.?

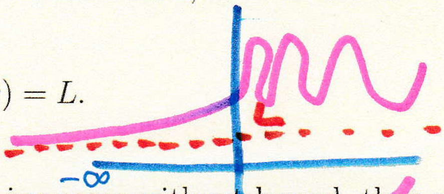
$\lim\limits_{x\to -2^+} \frac{x^2+4}{(x-2)(x+2)} = -\infty$ และ $\lim\limits_{x\to -2^-} \frac{x^2+4}{(x-2)(x+2)} = +\infty$ เป็น $\infty/-\infty$

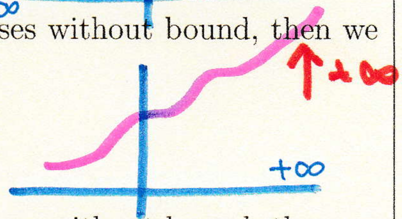
Limits at Infinity ลิมิตที่อนันต์

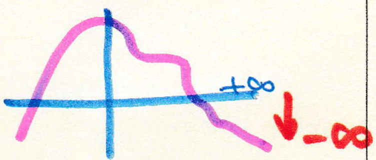


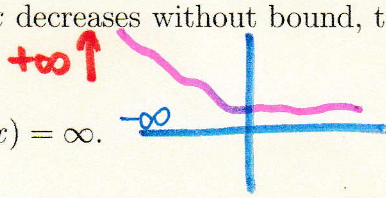
Let L be any real number.

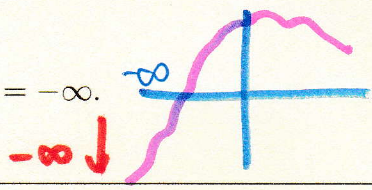
1. If $f(x)$ approaches L as x increases without bound, then we write
$$\lim_{x\to\infty} f(x) = L.$$


2. If $f(x)$ approaches L as x decreases without bound, then we write
$$\lim_{x\to-\infty} f(x) = L.$$


3. If $f(x)$ increases without bound as x increases without bound, then we write
$$\lim_{x\to\infty} f(x) = \infty.$$


4. If $f(x)$ decreases without bound as x increases without bound, then we write
$$\lim_{x\to\infty} f(x) = -\infty.$$


5. If $f(x)$ increases without bound as x decreases without bound, then we write
$$\lim_{x\to-\infty} f(x) = \infty.$$


6. If $f(x)$ decreases without bound as x decreases without bound, then we write
$$\lim_{x\to-\infty} f(x) = -\infty.$$


> Let k be any real number except 0 and n a positive real number.
>
> 1. $\lim\limits_{x\to-\infty} k = k = \lim\limits_{x\to\infty} k$
> 2. $\lim\limits_{x\to\infty} \dfrac{k}{x^n} = 0 = \lim\limits_{x\to-\infty} \dfrac{k}{x^n}$
> 3. $\lim\limits_{x\to\infty} kx^n = \begin{cases} \infty & \text{if } k > 0, \\ -\infty & \text{if } k < 0, \end{cases}$
> 4. $\lim\limits_{x\to-\infty} kx^n = \begin{cases} \infty & \text{if } k > 0, n \text{ is even}, \\ -\infty & \text{if } k > 0, n \text{ is odd}, \\ -\infty & \text{if } k < 0, n \text{ even}, \\ \infty & \text{if } k < 0, n \text{ odd}. \end{cases}$

Example 13. Find each limit.

1. $\lim\limits_{x\to\infty} 4 =$ 4
2. $\lim\limits_{x\to-\infty} 3 =$ 3
3. $\lim\limits_{x\to-\infty} \dfrac{1}{x^4} =$ 0
4. $\lim\limits_{x\to\infty} \dfrac{1}{x^2} + \dfrac{3}{x} =$ 0
5. $\lim\limits_{x\to\infty} 2x^5 =$ $+\infty$
6. $\lim\limits_{x\to-\infty} 2x^5 =$ $-\infty$
7. $\lim\limits_{x\to\infty} -7x^6 =$ $-\infty$
8. $\lim\limits_{x\to-\infty} -7x^6 =$ $-\infty$

> **Limits of Polynomial Functions at Infinity**
>
> If $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0, \qquad a_n \neq 0, \quad n \geq 1$, then
>
> $$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} a_n x^n \text{ and } \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} a_n x^n.$$

$\cancel{\infty - \infty = 0}$ อนิยาม
indeterminate form.

Example 14. Find limits of the following functions.

1. $\lim\limits_{x\to\infty} (2x^5 + 5)$ $= +\infty$
2. $\lim\limits_{x\to-\infty} (2x^5 - 5)$ $= -\infty$
3. $\lim\limits_{x\to\infty} (\overset{-\infty}{-7x^6} + \overset{+\infty}{2x^5})$ $= \lim\limits_{x\to\infty} x^6\left(-7 + \dfrac{2}{\cancel{x}}\right)$ $= -\infty$
4. $\lim\limits_{x\to\infty} (7x^6 + 2x^5)$ $= +\infty$