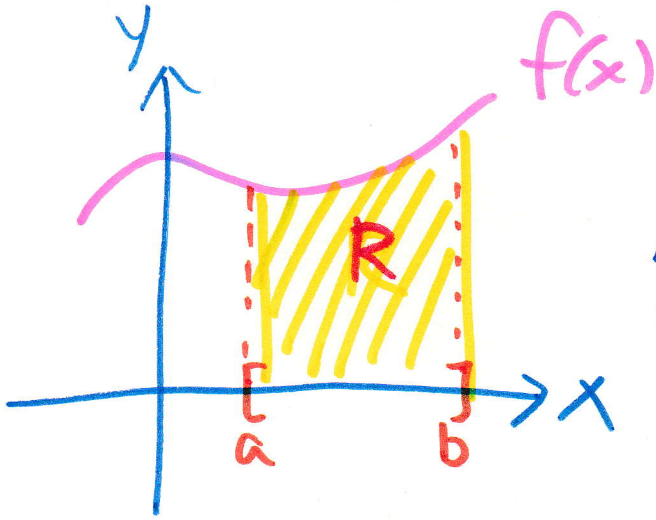
ทบทวน

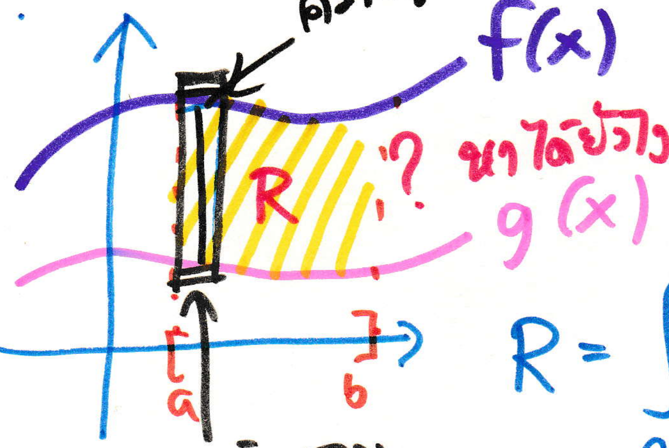


Area under $f(x)$ from a to b on $[a,b]$ คือ

$$A=\int_a^b f(x)\,dx.$$

ความสูงของ $\square = f(x^*)-g(x^*)$

Q.



$f(x) \geq g(x)\ \forall x \in [a,b].$

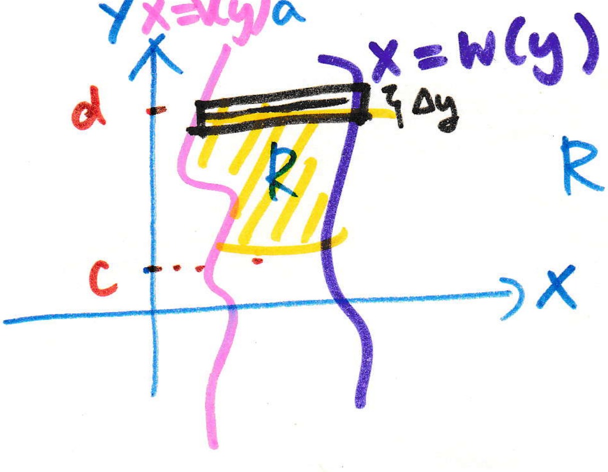
$$R=\int_a^b [f(x)-g(x)]\,dx$$

ความกว้างแถบ

$\Delta x_k \to$ มองว่า dx เล็กๆ

$$\text{Area} = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \underbrace{[f(x_k^*)-g(x_k^*)]}_{\text{ความสูงของ}\ \square}\ \underbrace{\Delta x_k}_{\text{ความกว้าง}}$$

$$= \int_a^b [\boxed{f(x)} - g(x)]\,dx.$$

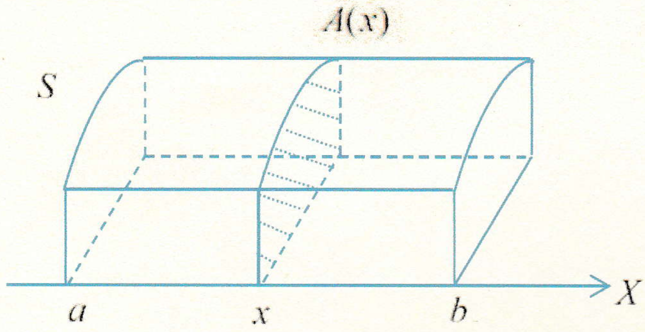


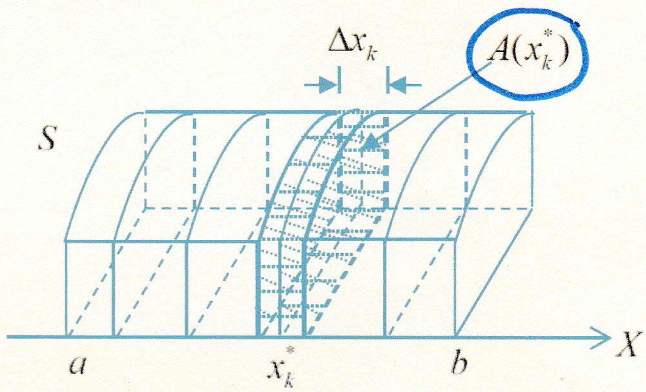
$$R=\int_c^d [w(y)-v(y)]\,dy$$

6.2 Volumes by slicing; Disks and Washers

Theorem 6.3 (**Volume formula**) Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in [a, b], the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$\boxed{V = \int_a^b \underbrace{A(x)}_{\text{พ.ท.ฐาน}}\, \underset{\text{ความสูง}}{dx}.} \tag{6.3}$$



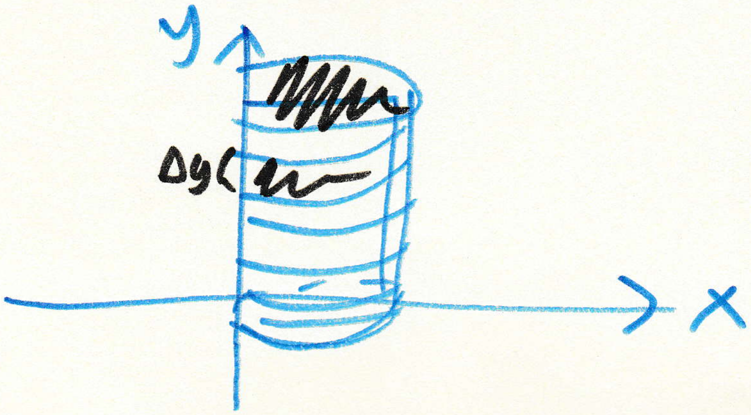


พ.ท.ผิว ความกว้าง.
ของชิ้นย่อย

$$V = \lim_{max \Delta x_k \to 0} \underbrace{\sum_{k=1}^{n} \boxed{A(x_k^*)} \Delta x_k} = \boxed{\int_a^b A(x)dx}$$

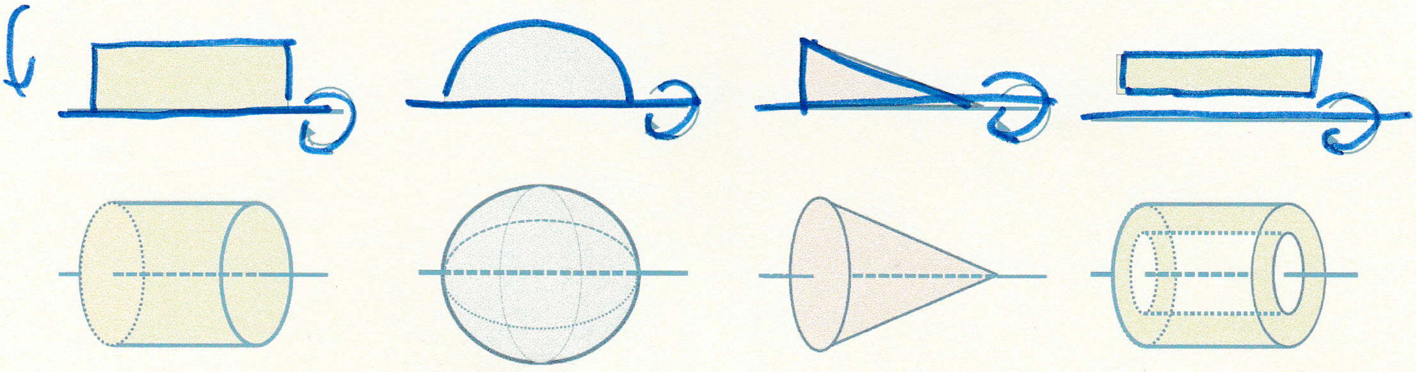
There is a similar result for cross sections perpendicular to the y-axis.

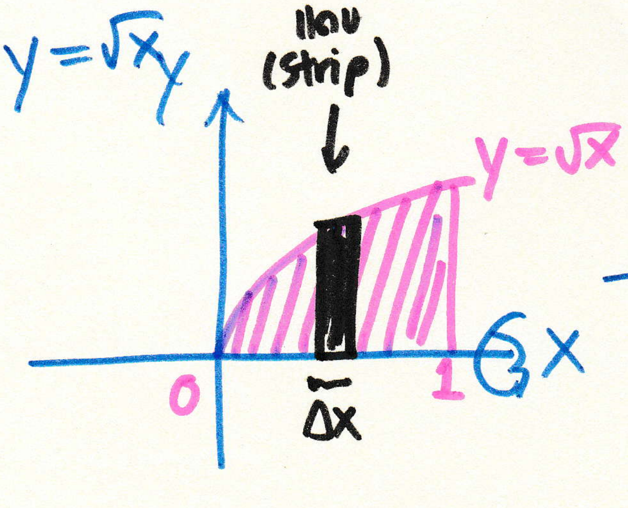
> **Theorem 6.4 (Volume formula)** Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in [c, d], the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is
>
> $$V = \int_c^d A(y)dy. \tag{6.4}$$



Solid of revolution ทรงตันที่เกิดจากการหมุนรอบแกน

Rotation axis : แกนหมุน





หาปริมาตรของทรงตันที่เกิดจาก
การหมุนรอบแกน X

→ หน้าตัดเป็นรูป วงกลม

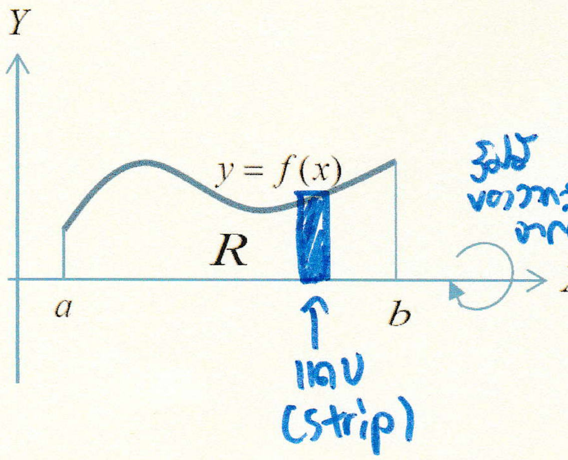
*ความสูงของแถบ
เป็นรัศมีของวงกลมหน้าตัด

$\therefore V(1\ \text{ชิ้น}) = (\pi r^2)\Delta x$

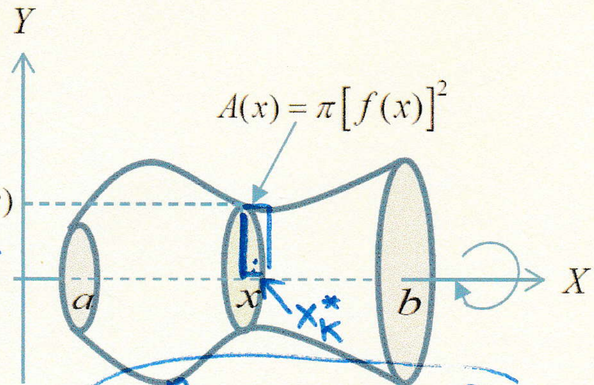
$V(1\ \text{ชิ้น}) = \pi(f(x))^2\Delta x$

Volume by Disks perpendicular to the X-axis

Problem: Let f be continuous and nonnegative on [a, b], and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$. Find the volume of the solid of revolution that is generated by revolving the region R about the X-axis.



รัศมีของวงกลมจากการหมุน.



$$V = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \pi (f(x_k^*))^2 \Delta x_k$$

(disk method
ทรงหมุนแบบจาน.) $\longrightarrow$ $\boxed{V = \int_a^b \underbrace{\pi (f(x))^2}_{\text{พื้นที่ area}} \underbrace{dx}_{\text{ความกว้าง}}}$

We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the X-axis at the point x is a circular disk of radius $f(x)$. The area of this region is

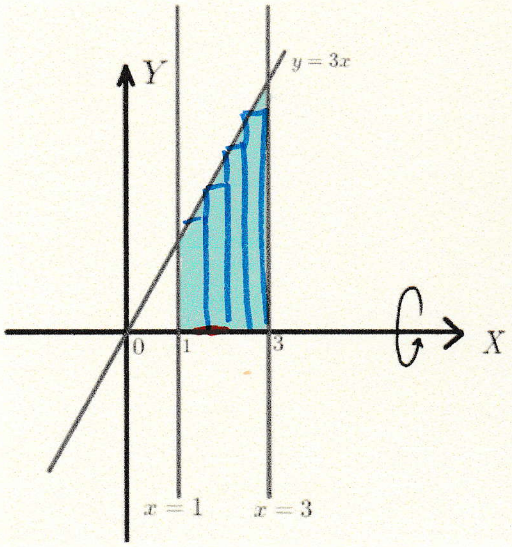
$$A(x) = \pi[f(x)]^2.$$

Thus, from (6.3) the volume of the solid is

$$V = \int_a^b \underbrace{\pi[f(x)]^2}_{\text{พ.ท.ฐาน}} \underbrace{dx}_{\text{กว้าง.}}. \tag{6.5}$$

Because the cross sections are disk shaped, the application of this formula is called the *method of disks.*

Example 6.6 Find the volume of the solid that is obtained when the region under the curve $y = 3x$ over the interval $[1, 3]$ is revolved about the $\boxed{X\text{-axis.}}$



$A(x) = \pi(3x)^2$

$$V = \int_1^3 \pi(3x)^2 dx$$

$$= 9\pi \int_1^3 x^2 dx$$

$$= 9\pi \left[\frac{x^3}{3}\right]_1^3$$

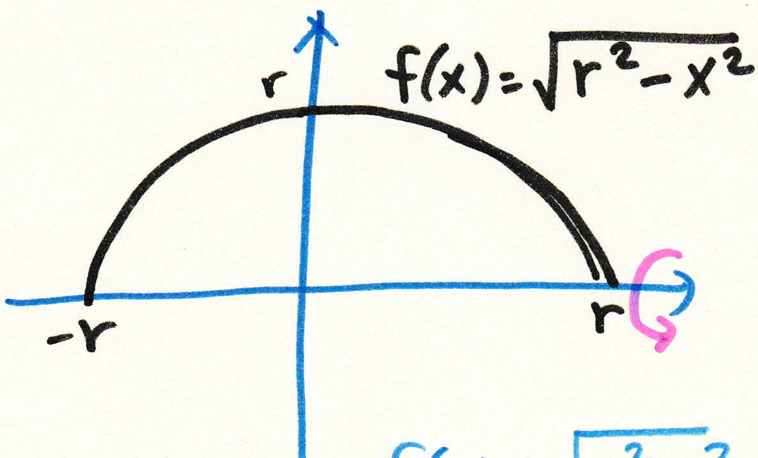
$$= 9\pi \left[\frac{27}{3} - \frac{1}{3}\right]$$

$$= \cancel{9}^{3}\pi \left[\frac{26}{\cancel{3}_1}\right] = 78\pi \quad \#$$

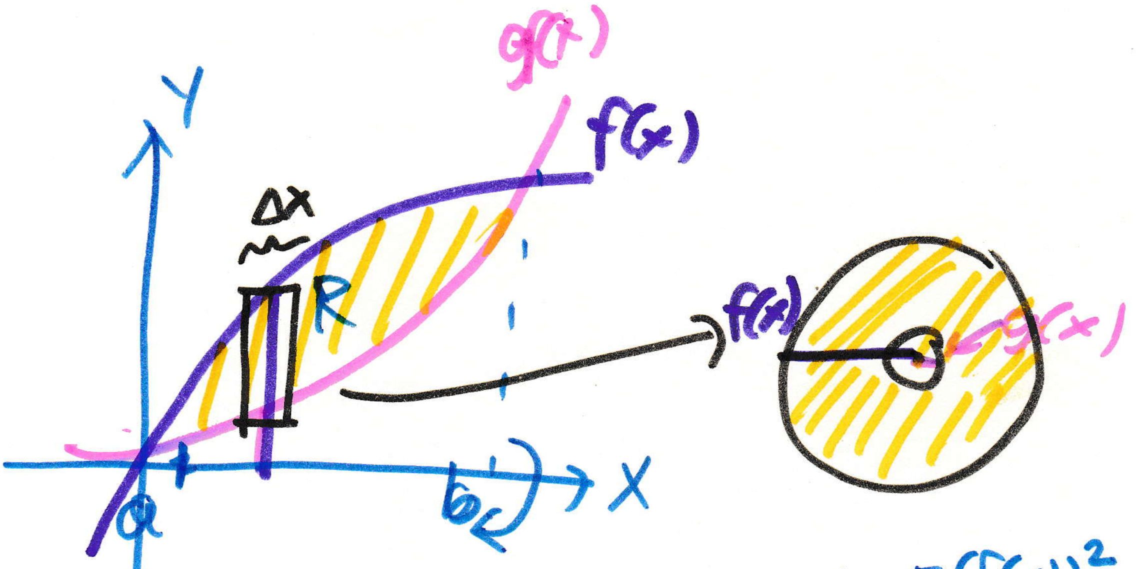
$\boxed{\text{โจทย์ท้าทายปัญญา}}$

จงพิสูจน์ว่า ปริมาตรของทรงกลมที่มีรัศมี r หาได้จาก $V = \frac{4}{3}\pi r^3$

$V = \frac{4}{3}\pi r^3$



หาปริมาตรทรงตันจากการหมุน $f(x) = \sqrt{r^2 - x^2}$ รอบแกน X
(จาก -r ถึง r : บน $[-r, r]$)



$$A(x) = \text{พ.ท.วงแหวน} = \underline{\pi(f(x))^2 - \pi(g(x))^2}$$

$$V = \int_a^b A(x)\,dx$$

รัศมีใหญ่ (f ตัวบน)

รัศมีเล็ก (f ตัวล่าง)

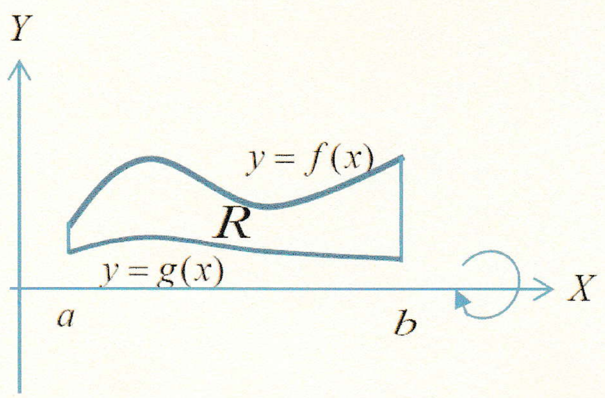
$$\boxed{V = \int_a^b \pi\Big[\underbrace{(f(x))^2 - (g(x))^2}_{\text{พ.ท.ผิว}}\Big]\underbrace{dx}_{\text{ความกว้างอันย่อย}}}$$

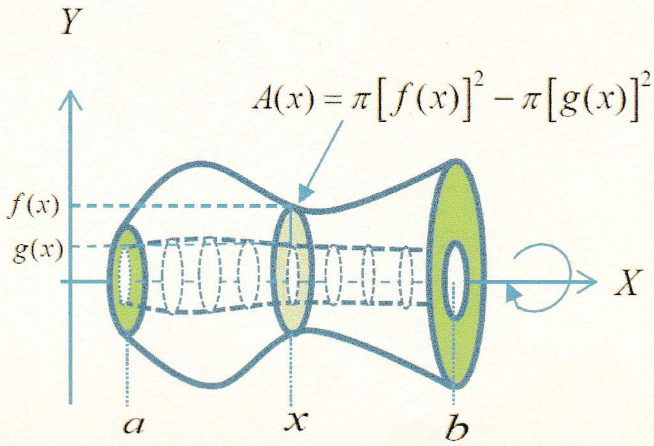
หน้าตัดเป็นวงแหวน ⇒ Washer Method

"Ring Method"

Volume by Washers perpendicular to the X-axis

Problem: Let f and g be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$. Find the volume of the solid of revolution that is generated by revolving the region R about the X-axis.





We can solve this problem by slicing. For this purpose, observe that the cross section of the solid taken perpendicular to the X-axis at the point x is the annular or "washer-shaped", region with inner radius $g(x)$ and outer radius $f(x)$. The area of this region is

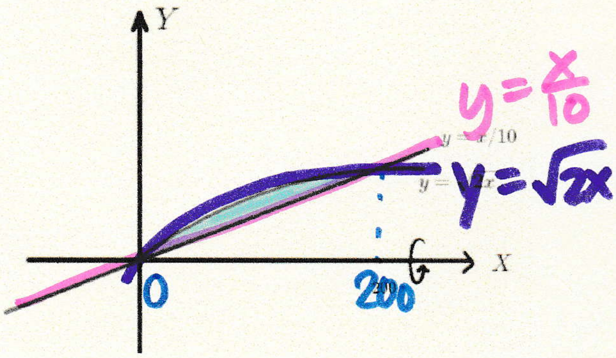
$$A(x) = \pi[f(x)]^2 - \pi[g(x)]^2 = \pi\left([f(x)]^2 - [g(x)]^2\right)$$

Thus, from (6.3) the volume of the solid is

$$V = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right)dx \tag{6.6}$$

Because the cross sections are washer shaped, the application of this formula is called the ***method of washers.***

Example 6.7 Find the volume of the solid that is obtained when the region between the graphs of the equations $y = \sqrt{2x}$ and $y = \frac{x}{10}$ over the interval $[0, 200]$ is revolved about the X-axis.



$$V = \pi \int_0^{200} \left[\left(\sqrt{2x}\right)^2 - \left(\frac{x}{10}\right)^2 \right] dx$$

$$= \pi \int_0^{200} \left[2x - \frac{x^2}{100} \right] dx$$

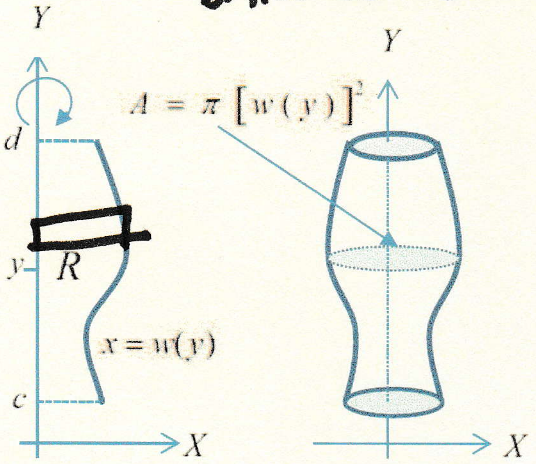
$$= \pi \left[x^2 - \frac{x^3}{300} \right]_0^{200}$$

$\vdots$

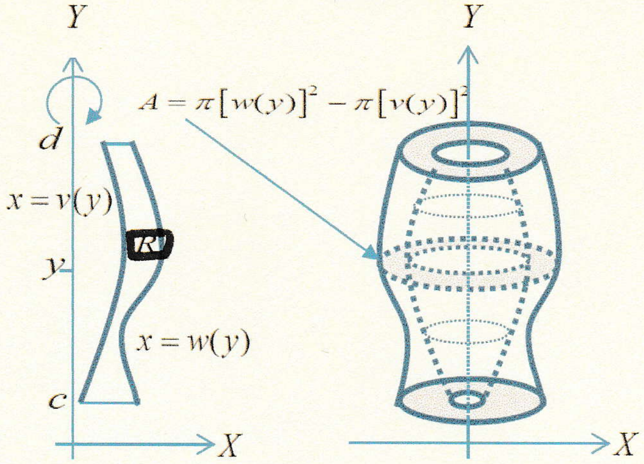
$$= \frac{\pi}{3}(40{,}000) \qquad \#$$

Volume by Disks and Washers perpendicular to the Y-axis

ถ้าสร้างแถบของ พ.ท. แถบจะตั้งฉากกับแกนหมุน.



Disks



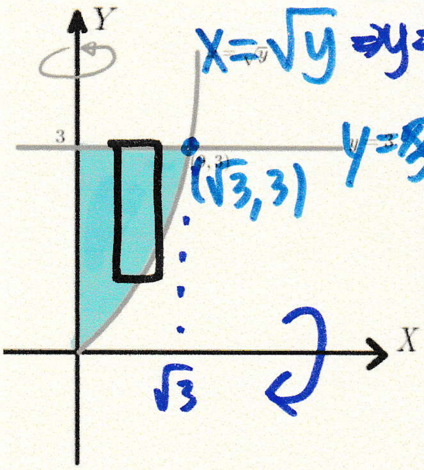
Washers

The methods of disks and washers have analogs for regions that are revolved about the Y-axis. Using the method of slicing and Formula (6.4), the following formulas for the volumes of the solid are

$$V = \int_c^d \pi [w(y)]^2 dy \qquad (disks), \tag{6.7}$$

$$V = \int_c^d \pi \left([w(y)]^2 - [v(y)]^2 \right) dy \qquad (washers). \tag{6.8}$$

Example 6.8 Find the volume of the solid generated when the region enclosed by $x = \sqrt{y}$, $x = 0$, and $y = 3$ is revolved about the Y-axis.



$x = \sqrt{y} \Rightarrow y = x^2$

$\boxed{\text{Disk}}$ $V = \int_0^3 \pi (\sqrt{y})^2 \, dy$

$= \pi \int_0^3 y \, dy$

$= \pi \left[\frac{y^2}{2} \right]_0^3$

$= \frac{9}{2} \pi$ #

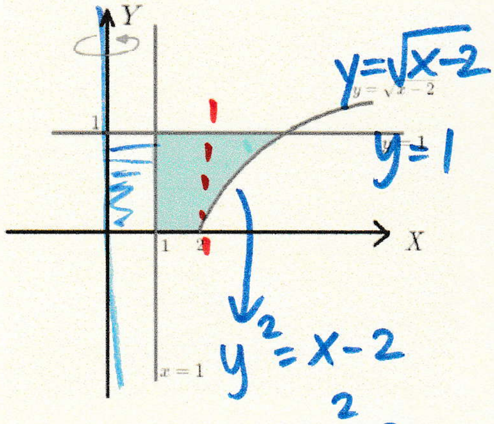
$\boxed{\text{แบบหมุนรอบแกน X}}$ ใช้ washer method

$V_x = \pi \int_0^{\sqrt{3}} \left[(3)^2 - (x^2)^2 \right] dx$

$= \pi \int_0^{\sqrt{3}} (9 - x^4) \, dx$

$= \pi \left[9x - \frac{x^5}{5} \right]_{x=0}^{\sqrt{3}} = \ldots\ldots = \pi \left[\frac{36}{5} \sqrt{3} \right]$ #

Example 6.9 Find the volume of the solid generated when the region enclosed by $x = 1$, $y = \sqrt{x-2}$, $y = 0$, and $y = 1$ is revolved about the Y-axis.



$y^2 = x - 2$

$\therefore x = y^2 + 2$

หมุนรอบแกน y ใช้ washer method

$$V = \pi \int_0^1 [(y^2+2)^2 - 1^2]\,dy$$

$$= \pi \int_0^1 [(y^4 + 4y^2 + 4) - 1]\,dy$$

$$= \pi \int_0^1 (y^4 + 4y^2 + 3)\,dy$$

$$= \pi \left[\frac{y^5}{5} + \frac{4y^3}{3} + 3y\right]_0^1$$

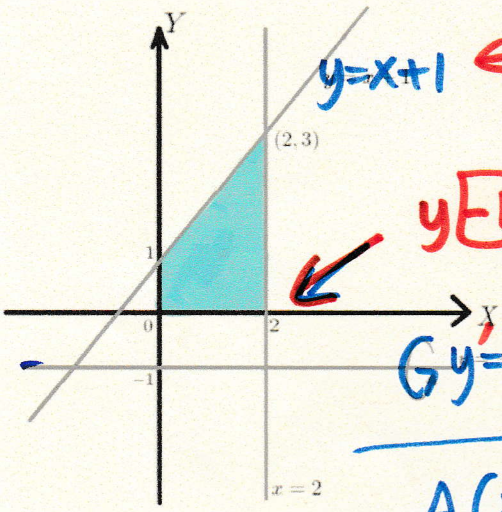
$$= \pi \left[\frac{1}{5} + \frac{4}{3} + 3\right] \quad \#$$

HW ลองหาปริมาตรเมื่อหมุนแกน X

Other axes of revolution

It is possible to use the method of disks and the method of washers to find the volume of a solid of revolution whose axis of revolution is a line other than one of the coordinate axes. Instead of developing a new formula for each situation, we will appeal to Formulas (6.3) and (6.4) and integrate an appropriate cross-sectional area to find the volume.

Example 6.10 Find the volume of the solid that is obtained when the region between the curve $y = x + 1$ and $y = 0$ over the interval $[0, 2]$ is rotated about the line $y = -1$.



$y = x+1 \leftarrow \boxed{y-1} = x+1 \leftarrow$ เลื่อนแกน

$y' = x+2$

$\boxed{y-1} = 0 \leftarrow$ เลื่อนแกน

$y' = 1$

$y' = -1 \leftarrow$ มองว่าเป็น $y=0$ อันใหม่

$$A(x) = \pi\left((x+2)^2 - (1)^2\right)$$

$$V = \int_0^2 \pi\left[(x+2)^2 - 1\right] dx$$

$$= \pi\int_0^2 \left((x^2+4x+4) - 1\right) dx$$

$$= \pi\int_0^2 (x^2+4x+3)\, dx$$

$$= \pi\left[\frac{x^3}{3} + \frac{4x^2}{2} + 3x\right]_0^2$$

$$= \pi\left[\frac{8}{3} + \frac{16}{2} + 6\right] = \pi\left[\frac{50}{3}\right].$$ #