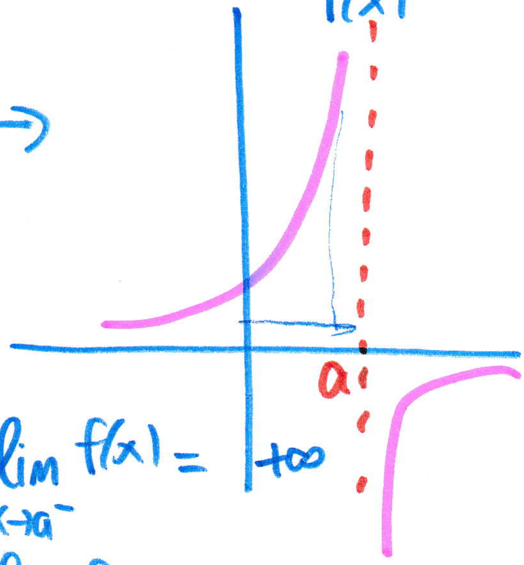
Limit ปกติ

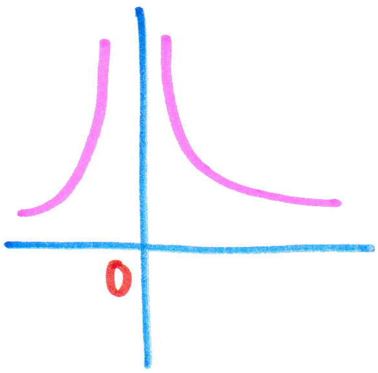
Limit อนันต์ (Infinity limit)



$$\lim_{x \to a^-} f(x) = +\infty$$

$$\lim_{x \to a^+} f(x) = -\infty$$

$$\lim_{x \to a} f(x) = \text{ไม่มี}$$

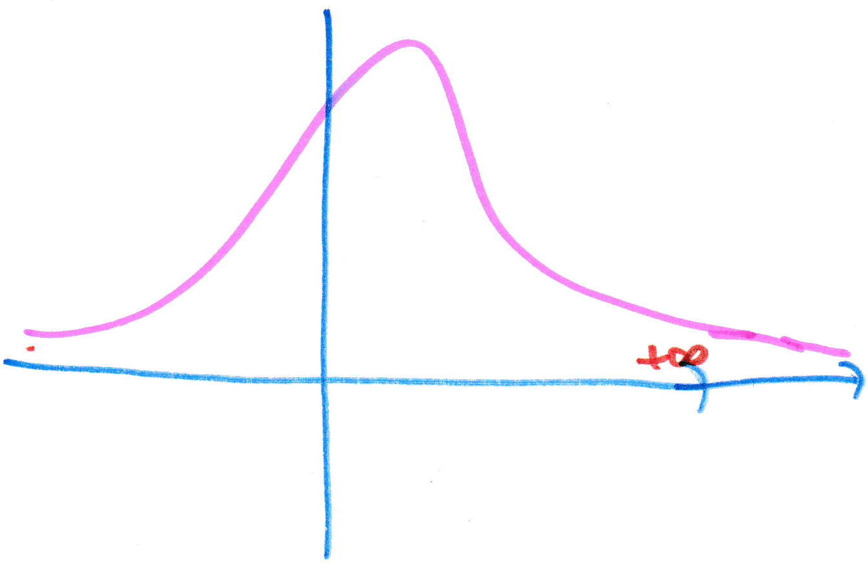


$$\lim_{x \to 0^+} f(x) = +\infty$$

$$\lim_{x \to 0^-} f(x) = +\infty$$

$$\lim_{x \to 0} f(x) = +\infty$$

Limit ที่อนันต์



$$\lim_{x \to +\infty} f(x) = 0$$

$$\lim_{x \to +\infty} f(x) = 0$$

ฟังก์ชันพหุนาม $\Rightarrow p(x) = x^5 + x^4 + 3x^3$

$$\lim_{x \to +\infty} p(x) = \infty$$

$$\Rightarrow p(x) = \underset{-\infty}{-7x^6} + \underset{+\infty}{x^5} - \underset{-\infty}{x^4} = x^6\left(-7 + \frac{x^5}{x^6} - \frac{x^4}{x^6}\right)$$

$$\lim_{x \to +\infty} p(x) = \cancel{\infty}$$

$$= \lim_{x \to +\infty} x^6\left(-7 + \cancelto{0}{\frac{1}{x}} - \cancelto{0}{\frac{1}{x^2}}\right)$$

$$= -\infty$$

ฟังก์ชันตรรกยะ

$$f(x) = \frac{n(x)}{d(x)}$$

$$f(x) = \frac{1}{x^2-4} \quad \boxed{D_f = \mathbb{R} - \{2, -2\}}$$

V.A: $x = -2, +2$

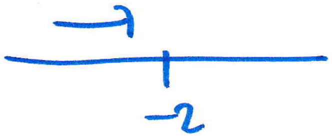
$$\lim_{x \to 2^-} f(x) = \cancel{+\infty}$$

$$\hookrightarrow \lim_{x \to 2^-} \frac{1}{(x-2)(x+2)} = -\infty$$

$$\lim_{x \to 2^+} \frac{1}{(x-2)(x+2)} = +\infty$$

$x=2$ เป็น V.A.

$$\lim_{x \to -2} \frac{1}{(x-2)(x+2)} \; ?$$



$$\lim_{x \to -2^-} \frac{1}{(x-2)(x+2)} = +\infty$$

$$\lim_{x \to -2^+} \frac{1}{(x-2)(x+2)} = -\infty$$

$x = -2$ เป็น V.A.

Limits of Rational Functions at Infinity

If $f(x) = \dfrac{n(x)}{d(x)}$ is a rational function defined by

$$n(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0,$$

$$d(x) = b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0,$$

where $a_n, b_m \neq 0$, then

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{a_n x^n}{b_m x^m} \text{ and } \lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{a_n x^n}{b_m x^m}$$

Example 15. Find limits of the following rational functions.

1. $\displaystyle \lim_{x\to\infty} \frac{3x+5}{6x-8}$
2. $\displaystyle \lim_{x\to-\infty} \frac{4x^2-x}{2x^3-5}$
3. $\displaystyle \lim_{x\to-\infty} \frac{5x^3-2x^2+1}{3x+5}$

(ดูใน กระดาษ)

Limits of Rational Functions at Infinity and Horizontal Asymptotes of Rational Functions

From the rational function $f(x) = \dfrac{n(x)}{d(x)}$ above, where n is the degree of $n(x)$ and m is the degree of $d(x)$,

1. If $m > n$, then $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = 0$, and the line $y = 0$ (the x axis) is a horizontal asymptote of $f(x)$.
2. If $m = n$, then $\lim_{x\to\infty} f(x) = \lim_{x\to-\infty} f(x) = \dfrac{a_n}{b_m}$, and the line $y = \dfrac{a_n}{b_m}$ is a horizontal asymptote of $f(x)$.
3. If $m < n$, then each limit will be ∞ or $-\infty$, depending on m, n, a_n, and b_m, and $f(x)$ does not have a horizontal asymptote.

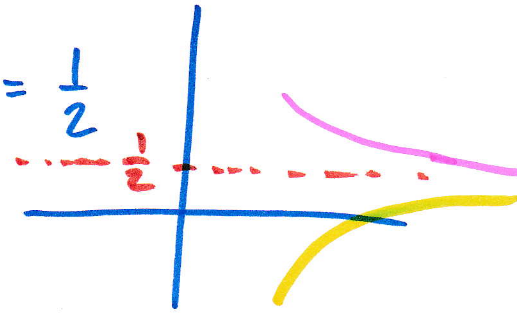
Horizontal Asymptote

(♥)

$$\boxed{\lim_{x\to+\infty} f(x) = L_1} \Rightarrow y = L_1 \text{ เป็น horizontal asymptote.}$$

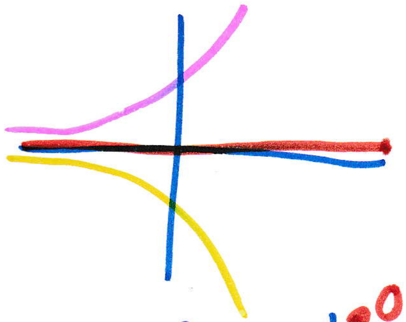
$$\boxed{\lim_{x\to-\infty} f(x) = L_2} \Rightarrow y = L_2 \text{ เป็น horizontal asymptote}$$

Ex 1 $$\lim_{x\to+\infty} \frac{3x+5}{6x-8} = \lim_{x\to+\infty} \frac{\frac{3\cancel{x}}{\cancel{x}} + \frac{5}{x}\to 0}{\frac{6\cancel{x}}{\cancel{x}} - \frac{8}{x}\to 0} = \frac{3}{6} = \frac{1}{2}$$



นำเทอมที่กำลังสูงสุด $(x^?)$ ของตัวส่วน ไปหารทุกเทอม.

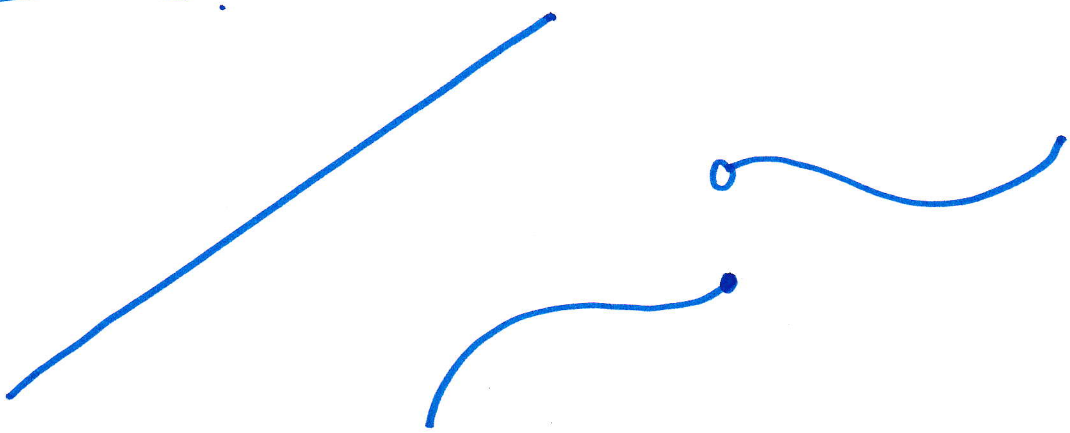
Ex 2 $$\lim_{x\to-\infty} \frac{4x^2-x}{2x^3-5} = \lim_{x\to-\infty} \frac{\frac{4x^2}{x^3} - \frac{x}{x^3}\to 0}{\frac{2\cancel{x^3}}{\cancel{x^3}} - \frac{5}{x^3}\to 0} = 0$$

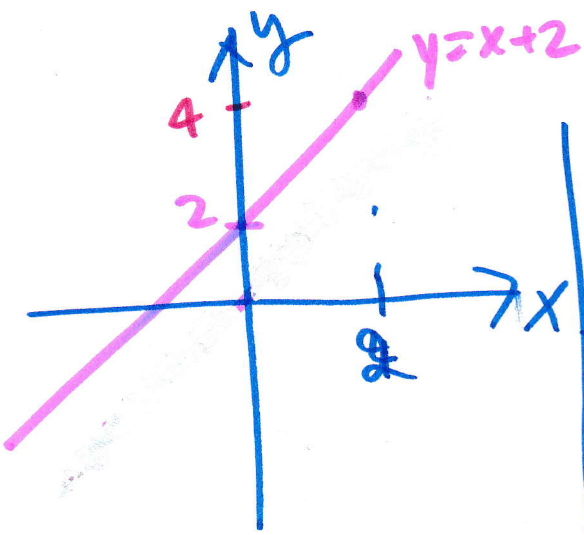


Ex 3 $$\lim_{x\to-\infty} \frac{5x^3-2x^2+1}{3x+5} = \lim_{x\to+\infty} \frac{\frac{5x^3}{x} - \frac{2x^2}{x} + \frac{1}{x}}{\frac{3\cancel{x}}{\cancel{x}} + \frac{5}{x}\to(0)} = \lim_{x\to-\infty} \frac{5x^2-2x+\frac{1}{x}\to 0}{3+\frac{5}{x}\to 0}$$

$$= +\infty$$

ความต่อเนื่อง



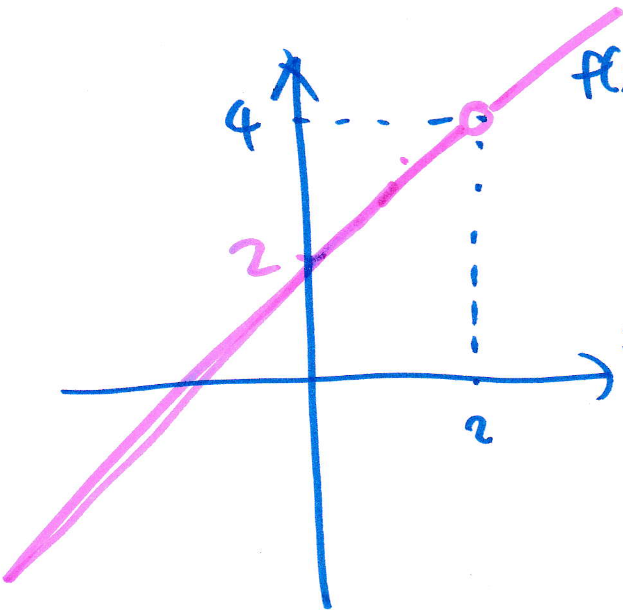


ที่ $x=2$ $(f(x)=x+2)$

$f(2) = 4$

$\lim_{x \to 2} f(x) = 4$

$$\boxed{\lim_{x \to 2} f(x) = f(2) = 4}$$



$f(x) = \frac{x^2-4}{x-2}$ $\quad D_f = \mathbb{R} - \{2\}$

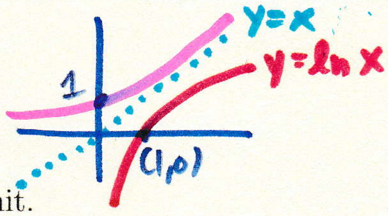
$f(2)$ หาค่าไม่ได้

$\lim_{x \to 2} f(x) = 4$

$\therefore f$ ต่อเนื่องที่ $x=2$ ไหม?

NO

$y = e^x$



Example 16. Find each limit.

2^{-2x} $(2^{-2})^x = (\frac{1}{4})^x$

1. $\lim_{x\to\infty} \log_3 x$ $= +\infty$

   $\lim_{x\to 0^+} \log_3 x = -\infty$ (1,0)

2. $\lim_{x\to\infty} e^{-2x}$ $= \lim_{x\to\infty} (e^{-2})^x = \lim_{x\to\infty} (\frac{1}{e^2})^x$

   $y = e^{2x}$, $y = e^{-2x}$

   $\therefore \lim_{x\to+\infty} e^{-2x} = 0$

3. $\lim_{x\to\infty} \ln(7x)$ H.A: $y=0$

   $y = \ln(7x)$

   $\lim_{x\to\infty} \ln(7x) = +\infty$

4. $\lim_{x\to\infty} 2^{-2x}$ $= 0$

   $y = 2^{-2x}$

   $\lim_{x\to-\infty} 2^{-2x} = +\infty$

5. $\lim_{x\to-\infty} (\frac{e}{3})^{5x}$ $= \lim_{x\to-\infty} (\frac{e}{3})^{5x}$

   $y = (\frac{e}{3})^{5x}$

   $\lim_{x\to-\infty} (\frac{e}{3})^{5x} = +\infty$

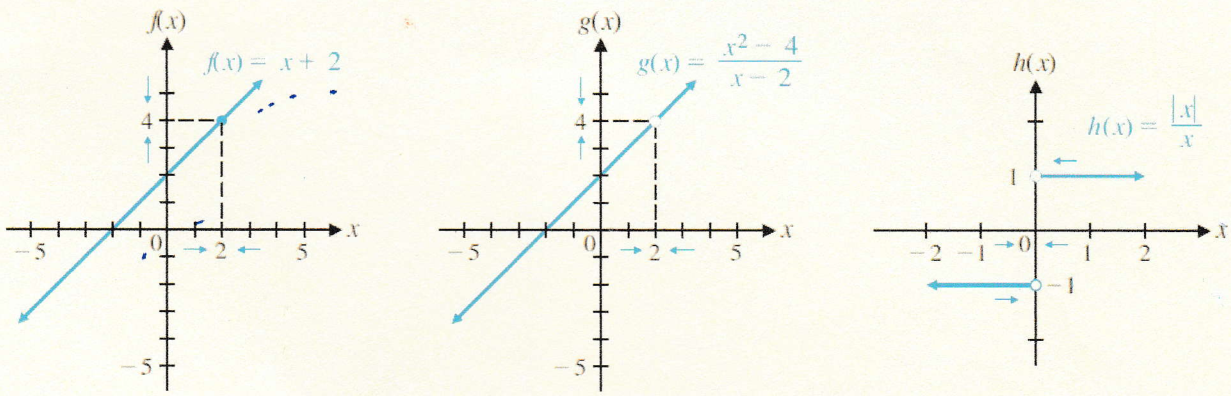
6. $\lim_{x\to\infty} \ln(4) + e^7$ $= \ln(4) + e^7$

   $f(x) = \ln(x) \Rightarrow f(4) = \ln(4)$

4.3 Continuity

ความต่อเนื่อง

> **Continuity**
>
> A function f is **continuous** at the point $x = a$ if f satisfies the following 3 properties.
>
> 1. $f(a)$ exists 2. $\lim_{x\to a} f(x)$ exists 3. $\lim_{x\to a} f(x) = f(a)$
>
> If one or more of the three conditions above fails, then the function f is **discontinuous** at $x = a$.



(A) $\lim_{x\to 2} f(x) = 4$, $f(2) = 4$

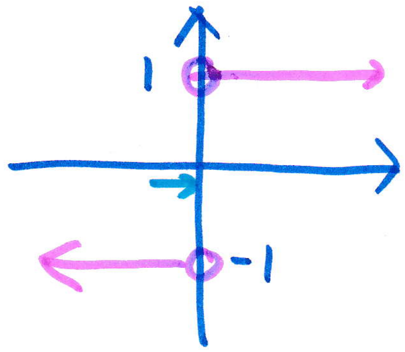
(B) $\lim_{x\to 2} g(x) = 4$, $g(2)$ is not defined

(C) $\lim_{x\to 0} h(x)$ does not exist, $h(0)$ is not defined

$$f(x)=\frac{|x|}{x} \qquad |x|=\begin{cases} x & ; x\geq 0 \\ -x & ; x<0 \end{cases}$$

$$f(x)=\begin{cases} \frac{x}{x} & ; x>0 \\ \frac{-x}{x} & ; x<0 \end{cases}$$

$$f(x)=\begin{cases} 1 & ; x>0 \\ -1 & ; x<0 \end{cases}$$



① $f(0)$ ไม่นิยาม

② $\lim\limits_{x\to 0} f(x)$ ไม่มี

$$\left.\begin{aligned} \lim_{x\to 0^-} f(x) &= -1 \\ \lim_{x\to 0^+} f(x) &= 1 \end{aligned}\right\}$$

$\therefore$ f ไม่ต่อเนื่องที่ $x=0$

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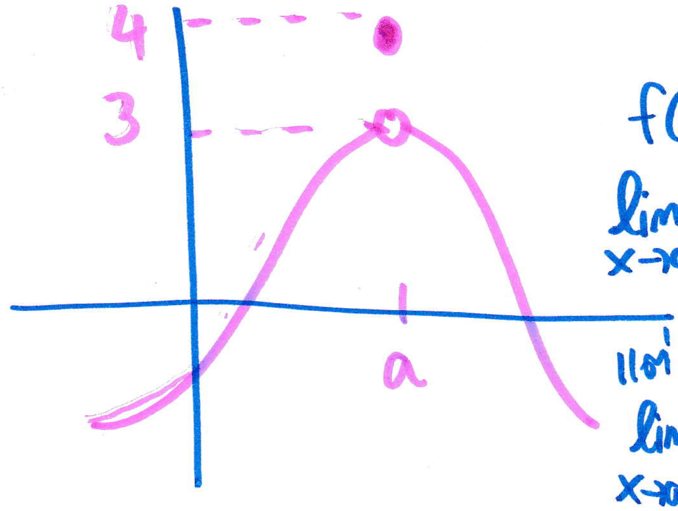
f เป็นฟังก์ชันต่อเนื่อง ที่ $\underline{x=a}$ $\Longleftrightarrow$ ① $f(a)$ หาค่าได้

② $\lim\limits_{x\to a} f(x)$ มีค่า

③ $\lim\limits_{x\to a} f(x) = f(a)$



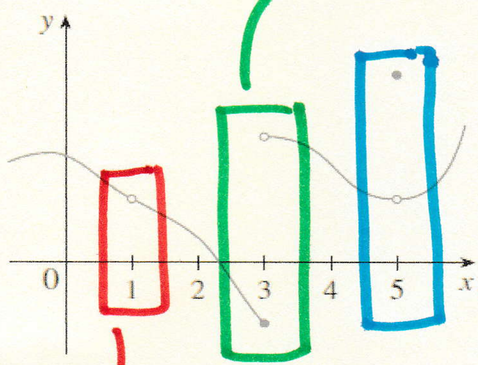
$f(a)=4$

$\lim\limits_{x\to a} f(x)=3$

แต่

$\lim\limits_{x\to a} f(x)=3\neq 4=f(a)$

Example 17. Given the graph of a function $f(x)$ as the following figure, at which point is $f(x)$ discontinuous? Explain why? Where is $f(x)$ continuous?



$x = 3$ $f(3)$ มี

$\lim_{x \to 3} f(x)$ ไม่มี

$\therefore f$ ไม่ต่อเนื่องที่ $x = 3$

$x = 5$ $f(5)$ มี

$\lim_{x \to 5} f(x)$ มี

$\therefore \lim_{x \to 5} f(x) \neq f(5)$

$\therefore f$ ไม่ต่อเนื่องที่ $x = 5$

$x = 1$ $\lim_{x \to 1} f(x)$ มี

$f(1)$ ไม่มี

$\therefore f$ ไม่ต่อเนื่องที่ $x = 1$

f cont at $x=a$ $\Leftrightarrow$ (1) $f(a)$ มี (2) $\lim_{x\to a} f(x)$ มี (3) $\lim_{x\to a} f(x) = f(a)$

Example 18. Let $f(x)$ be a function. Determine if $f(x)$ is continuous at the given point.

$D_f = \mathbb{R} - \{2\}$

1. $f(x) = \dfrac{x^2 - x - 2}{x - 2}$ at $x=2$

$f(2)$ ไม่มีค่า $\therefore$ f ไม่ต่อเนื่องที่ $x = 2$.

2. $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$ at $x = 0$

(1) $f(0) = 1$

(2) $\lim_{x\to 0} f(x) \rightarrow \left.\begin{array}{l} \lim_{x\to 0^-} f(x) = +\infty \\ \lim_{x\to 0^+} f(x) = +\infty \end{array}\right\}$ $\lim_{x\to 0} f(x) = +\infty$ ไม่มีค่าจำกัด

$\therefore$ f ไม่ต่อเนื่องที่ $x = 0$

3. $f(x) = \dfrac{|x|}{x}$ at $x = -1$

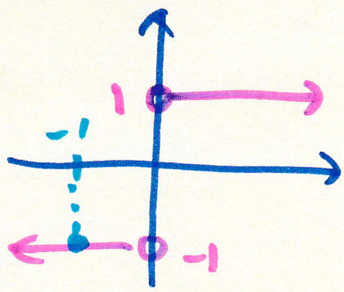
$f(x) = \begin{cases} 1 \ ; & x > 0 \\ -1 \ ; & x < 0 \end{cases}$

(1) $f(-1) = -1$

(2) $\lim_{x\to -1} f(x) = -1$

(3) $\lim_{x\to -1} f(x) = -1 = f(-1)$

$\therefore$ f cont ที่ $x = -1$



Example 19. Find the value of k so that the function $f(x) = \begin{cases} \dfrac{x^2-4}{x-2} & \text{if } x \neq 2 \\ k & \text{if } x = 2 \end{cases}$ is continuous.

f cont $\Leftrightarrow$ (1) $f(2) = k$

(2) $\lim_{x\to 2} \dfrac{x^2-4}{x-2} = \lim_{x\to 2} \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = \lim_{x\to 2}(x+2) = 4$

(3) $f(2) = \lim_{x\to 2} \dfrac{x^2-4}{x-2}$

จากผลลัพธ์ (1) = (2) นั่นคือ $\boxed{k=4}$