ทฤษฎีบท

(1) $\frac{d}{dx} c = 0$

(2) $\frac{d x^n}{dx} = n x^{n-1}$

(3) $\frac{d}{dx}(kx) = k\frac{dx}{dx} = k$

(4) $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

(5) $\frac{d}{dx}(u \pm v) = \frac{du}{dx} \pm \frac{dv}{dx}$

(6) $\frac{d(uv)}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

(7) $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{x^2}$

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(8) $\frac{d}{dx}(a^x) = a^x \ln a \longleftrightarrow \frac{d}{dx}(e^x) = e^x$

(9) $\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a} \longleftrightarrow \frac{d}{dx}(\ln x) = \frac{1}{x}$

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(10) $\frac{d}{dx}(\sin x) = \cos x$ (12) $\frac{d}{dx}(\cos x) = -\sin x$

(13) $\frac{d}{dx}(\tan x) = \sec^2 x$ (14) $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$

(15) $\frac{d}{dx}(\sec x) = \sec x \tan x$ (16) $\frac{d}{dx}(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x$

HW

(1) y' เมื่อ $y = 2\cos x - \log_2 x$

$$y' = 2(-\sin x) - \frac{1}{x\ln 2}$$

$2-\sin x$

(2) $g(x) = (x^2-5)e^x$

$$y' = (x^2-5)\frac{d}{dx}(e^x) + e^x\frac{d}{dx}(x^2-5)$$

$$= (x^2-5)e^x + e^x(2x)$$

(3) $\frac{dy}{dx}$ เมื่อ $y = \frac{\sin x\cos x}{x^7+7}$

$$\frac{dy}{dx} = \frac{(x^7+7)\frac{d}{dx}(\sin x\cos x) - \sin x\cos x\frac{d}{dx}(x^7+7)}{(x^7+7)^2}$$

$$= \frac{(x^7+7)\left[\sin x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(\sin x)\right] - (\sin x\cos x)(7x^6)}{(x^7+7)^2}$$

$$= \frac{(x^7+7)\left[(\sin x)(-\sin x) + (\cos x)(\cos x)\right] - 7x^6\sin x\cos x}{(x^7+7)^2}$$

$$= \frac{(x^7+7)\left[\cos^2 x - \sin^2 x\right] - 7x^6\sin x\cos x}{(x^7+7)^2} \quad \#$$

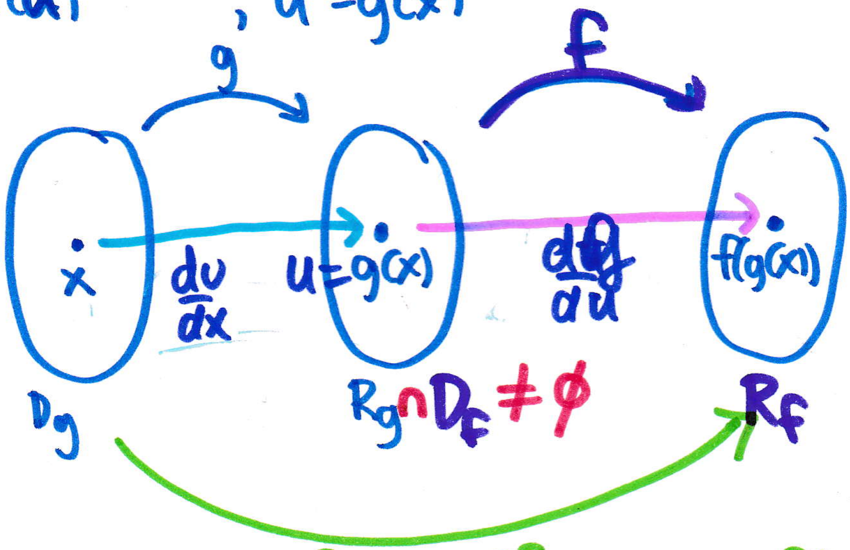
Composition of function

$y = f(x)$ $\quad u = g(x)$

ให้ $f(x) = x^2$, $g(x) = \dfrac{1}{x}$

$$f(\boxed{g(x)}) = f\left(\boxed{\frac{1}{x}}\right) = \left(\boxed{\frac{1}{x}}\right)^2$$

ให้ $y = f(x)$, $u = g(x)$



$f \circ g \Rightarrow (f \circ g)(x) = f(g(x))$

$$D_{f \circ g} = \{x \mid x \in D_g \text{ and } g(x) \in D_f\}$$

$$\therefore \boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

กฎลูกโซ่ (chain rule)

$$(f \circ g)'(x) = f'(g(x))\,g'(x)$$

5.7 The Chain Rule

A composite of functions $y = f(u)$ and $u = g(x)$ is

$$(f \circ g)(x) = f[g(x)].$$

Domain of $f \circ g$ (denoted by $D_{f \circ g}$) $= \Big\{ x \mid x \in D_g \quad \text{and} \quad g(x) \in D_f \Big\}$.

Example 5.7.1. Find $f[g(x)]$ for

1. $f(u) = u^3; \quad g(x) = 2x^3 + x + 5$

   $f(g(x)) = f(2x^3+x+5) = (2x^3+x+5)^3$

2. $f(u) = \sin u; \quad g(x) = 3x + 4$

   $f(g(x)) = f(3x+4) = \sin(3x+4)$

3. $f(u) = e^u; \quad g(x) = -4x^2$

   $f(g(x)) = e^{-4x^2}$

   หา $g[f(x)]$

   หา $g(e^x) = -4(e^x)^2$

Chain Rule

If $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Example 5.7.2. From the previous example, use the chain rule to find $\dfrac{dy}{dx}$ for

1. $y = u^3; \quad u = 2x^3 + x + 5$

   $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

   $= (3u^2)(6x^2+1)$

   $= 3(2x^3+x+5)^2(6x^2+1)$

2. $y = \sin u; \quad u = 3x + 4$

   $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$

   $= (\cos u) \cdot (3)$

   $= (\cos(3x+4))3$

3. $y = e^u; \quad u = -4x^2$, $x = \cos m$

หา $\frac{dy}{dm} = \frac{dy}{du} \cdot \frac{du}{dx} \cdot \frac{dx}{dm}$

$= e^u \cdot (-8x) \cdot (-\sin m)$

$= e^{-4x^2}(-8\cos m)(-\sin m) = 8e^{-4(\cos m)^2}(\sin m \cos m)$. #

$$\frac{dy}{dx} = (2u)\cdot(2)\left(-\frac{1}{(x+1)^2}\right)$$

$$= \left[2(2v+1)\right](2)\left(-\frac{1}{(x+1)^2}\right)$$

$$\boxed{\frac{dy}{dx} = \left[2\left(2\left(\frac{1}{x+1}\right)+1\right)\right](2)\left(-\frac{1}{(x+1)^2}\right)}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \underline{\left[2\left(\cancel{2}\left(\frac{1}{\cancel{2}}\right)+1\right)\right]}\,\underline{(2)}\left(-\frac{1}{2^2}\right)$$

$$= (4)(2)\left(-\frac{1}{4}\right) = -2$$

#

$\boxed{x=1}$

Example 5.7.3. Let $y = u^2 + 5,\ u = 2v + 1$ and $\boxed{v = \dfrac{1}{x+1}}$. Find $\left.\dfrac{dy}{dx}\right|_{x=1}$.

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dv}\cdot\frac{dv}{dx}$$

$$= (2u)\cdot(2)\cdot\left(-(x+1)^{-2}\right)$$

$v = (x+1)^{-1}$

$\dfrac{dv}{dx} = -(x+1)^{-2}\dfrac{d}{dx}(x+1)$

สมว่า $v = \dfrac{1}{X}$ $X = x+1$

$\dfrac{dv}{dx} = -(x+1)^{-2}$

เมื่อ $x = 1 \Rightarrow v = \frac{1}{2} \Rightarrow u = 2\left(\frac{1}{2}\right)+1$

$= 2$

$$\boxed{\left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{dy}{du}\right|_{u=2}\cdot\left.\frac{du}{dv}\right|_{v=\frac{1}{2}}\cdot\left.\frac{dv}{dx}\right|_{x=1}} = \cancel{4}\cdot 2\cdot\left(-\frac{1}{\cancel{4}}\right) = -2$$

\#

From the above example, we have the following rules.

General Derivative Rule

Let $u = u(x)$ be a differentiable function.

General Power Rule

$$\frac{d}{dx}[u^n] = nu^{n-1}\frac{du}{dx}$$

General Derivatives of Exponential and Logarithmic Functions

$$\frac{d}{dx}[\log_a u] = \frac{1}{u\ln a}\cdot\frac{du}{dx} \qquad \frac{d}{dx}[\ln u] = \frac{1}{u}\cdot\frac{du}{dx}$$

$$\frac{d}{dx}[a^u] = a^u\ln a\cdot\frac{du}{dx} \qquad \frac{d}{dx}[e^u] = e^u\cdot\frac{du}{dx}$$

General Derivatives of Trigonometric Functions

$$\frac{d}{dx}[\sin u] = \cos u\frac{du}{dx} \qquad \frac{d}{dx}[\cos u] = -\sin u\frac{du}{dx}$$

$$\frac{d}{dx}[\tan u] = \sec^2 u\frac{du}{dx} \qquad \frac{d}{dx}[\cot u] = -\csc^2 u\frac{du}{dx}$$

$$\frac{d}{dx}[\sec u] = \sec u\tan u\frac{du}{dx} \qquad \frac{d}{dx}[\csc u] = -\csc u\cot u\frac{du}{dx}$$

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$$f(x) = e^{\cos(\sin(x^2+2))}$$

จะ $$f'(x) = e^{\cos(\sin(x^2+2))} \frac{d}{dx}(\cos(\sin(x^2+2)))$$

$$= e^{\cos(\sin(x^2+2))}(-\sin(\sin(x^2+2))) \frac{d}{dx}(\sin(x^2+2))$$

$$= e^{\cos(\sin(x^2+2))}(-\sin(\sin(x^2+2)))\cos(x^2+2)\frac{d}{dx}(x^2+2)$$

$$= e^{\cos(\sin(x^2+2))}(-\sin(\sin(x^2+2)))\cos(x^2+2)(2x) \quad \#$$

Example 5.7.4. Find $\frac{dy}{dx}$ for

1. $\boxed{y = (x^3 + 2x)^{37}}$

$$\frac{dy}{dx} = 37(x^3+2x)^{36}\frac{d}{dx}(x^3+2x)$$
$$= 37(x^3+2x)^{36}(3x^2+2) \quad \#$$

$y = u^{37}$

$u = (x^3+2x)$

$$\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}$$
$$= (37u^{36})\cdot(3x^2+2)$$
$$= 37(x^3+2x)^{36}\cdot(3x^2+2)$$

2. $y = \cos(2x^3 + 3x + 2)$

$$\frac{dy}{dx} = -\sin(2x^3+3x+2)\frac{d}{dx}(2x^3+3x+2)$$
$$= [\sin(2x^3+3x+2)](6x^2+3) \quad \#$$

Example 5.7.5. Find

1. $h'(x)$ if $h(x) = 3x^3 \boxed{\sin(3x)}$

2. $\frac{d}{dx}[\frac{e^{2x}}{x^2+4}]$

① $$h'(x) = (3x^3)\frac{d}{dx}(\sin(3x)) + \sin(3x)\frac{d}{dx}(3x^3)$$
$$= (3x^3)\cos(3x)\frac{d}{dx}(3x) + (\sin(3x))(9x^2)$$
$$= (3x^3)\cos(3x)\cdot(3) + 9x^2\sin(3x)$$
$$= 9x^3\cos(3x) + 9x^2\sin(3x) \quad \#$$

② $$\frac{d}{dx}\left[\frac{e^{2x}}{x^2+4}\right] = \frac{(x^2+4)\frac{d}{dx}(e^{2x}) - e^{2x}\frac{d}{dx}(x^2+4)}{(x^2+4)^2}$$
$$= \frac{(x^2+4)e^{2x}\frac{d}{dx}(2x) - e^{2x}(2x)}{(x^2+4)^2}$$
$$= \frac{2(x^2+4)e^{2x} - 2xe^{2x}}{(x^2+4)^2} \quad \#$$

$$\ln(w^2+5)^{3/2} = \frac{3}{2}\ln(w^2+5)$$

$$\therefore \frac{d}{dx}\left(\frac{3}{2}\ln(w^2+5)\right) = \frac{3}{2}\frac{1}{(w^2+5)}\frac{d}{dw}(w^2+5)$$

3. $\dfrac{dh}{dw}$ if $h(w) = \ln(w^2+5)^{3/2}$

$$\frac{dh}{dw} = \frac{1}{(w^2+5)^{3/2}}\frac{d}{dw}(w^2+5)^{3/2}$$

$$= \left(\frac{1}{(w^2+5)^{3/2}}\right)\left(\frac{3}{2}(w^2+5)^{\frac{1}{2}}\frac{d}{dw}(w^2+5)\right)$$

$$= \frac{1}{(w^2+5)^{3/2}}\cdot\left(\frac{3}{2}(w^2+5)^{\frac{1}{2}}\right)(2w)$$

#

$$= \frac{3}{2}(w^2+5)^{\frac{3}{2}-\frac{1}{2}}(2w)$$

$$= \frac{3}{2}\cdot\frac{1}{w^2+5}(2w)$$

$\neq$

4. $\dfrac{dh}{dw}$ if $h(w) = [\ln(w^2+5)]^{3/2}$

$$\frac{dh}{dw} = \frac{3}{2}(\ln(w^2+5))^{\frac{3}{2}-1}\frac{d}{dw}(\ln(w^2+5))$$

$$= \frac{3}{2}(\ln(w^2+5))^{\frac{1}{2}}\cdot\frac{1}{w^2+5}\frac{d}{dw}(w^2+5)$$

$$= \frac{3}{2}(\ln(w^2+5))^{\frac{1}{2}}\cdot\frac{1}{w^2+5}\cdot(2w)$$

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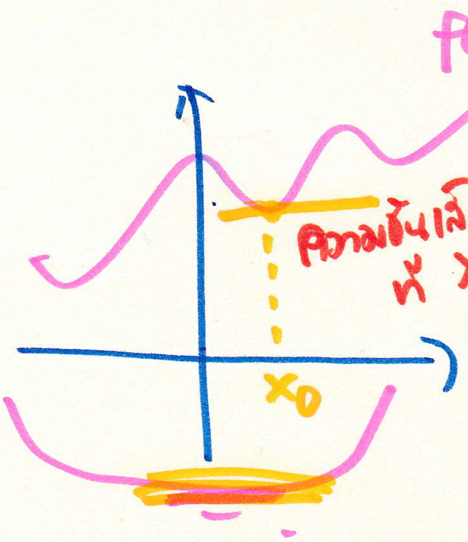
Example 5.7.6. Find the slope of the tangent line to the graph of $f(x) = 5e^{x^4+2x^3+1}$ at $x = 0$.



ความชันเส้นสัมผัสกราฟ $f(x)$ ที่ x_0 คือ ค่า $f'(x_0)$ → หา $f'(0)$

$$f'(x) = 5e^{x^4+2x^3+1}\frac{d}{dx}(x^4+2x^3+1)$$

$$= 5e^{x^4+2x^3+1}(4x^3+6x^2)$$

$$f'(0) = 5e^{0+0+1}(0+0)$$

$$= 5e^1(0) = 0$$