ทบทวน $\displaystyle\int 2^{\sin(\arctan 2x)} \frac{\cos(\arctan 2x)}{1+4x^2}\,dx \quad \text{—}(*)$

$$u = \sin(\arctan 2x)$$
$$du = \left(\cos(\arctan 2x)\frac{d}{dx}(\arctan 2x)\right)dx$$
$$= (\cos(\arctan 2x) \cdot \frac{1}{1+4x^2}\,(2)\,dx$$

$$(*) \doteq \frac{1}{2}\int \overbrace{2^u\,\frac{2\cos(\arctan(2x))}{1+4x^2}\,dx}^{du}$$

$$= \frac{1}{2}\int 2^u\,du \quad = \frac{1}{2}\frac{2^u}{\ln 2} + C$$

$$= \frac{1}{2}\cdot\frac{2^{\sin(\arctan 2x)}}{\ln 2} + C \quad \#$$

LIATE

$\displaystyle\int x^2 \sin x\,dx$ by parts

$u = x^2$ $\quad dv = \sin x$

$du = \sim\sim$ $\quad v = \sim\sim$

$$\int \overbrace{x^2 \sin(\overset{u}{x^3})\,dx}^{du}$$

เลือก $u = x^3$ $\quad du = 3x^2\,dx$

$$\int \frac{\arcsin x}{\sqrt{1-x^2}}\,dx$$

$\Rightarrow \displaystyle\int \arctan(2x)\,dx$

เลือก $u = \arctan$ $\Rightarrow u = \arctan 2x$

$dv = 2x\,dx$ $\Rightarrow dv = dx$

$$\int \sin^3 x \cos^2 x \, dx$$

$$= \int \sin^2 x \cdot \sin x \cdot \cos^2 x \, dx$$

$$= \int (1-\cos^2 x) \cdot \sin x \cdot \cos^2 x \, dx$$

$$= \int (\cos^2 x - \cos^4 x) \sin x \, dx$$

แทน $u = \cos x \quad \therefore du = -\sin x \, dx$

$\displaystyle\int \frac{1}{\sqrt{4-x^2}} dx$ ให้ $x = 2\sin\theta$

$dx = 2\cos\theta \, d\theta$

$$\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta}$$

$$= 2\sqrt{1-\sin^2\theta}$$

$$= 2\cos\theta$$

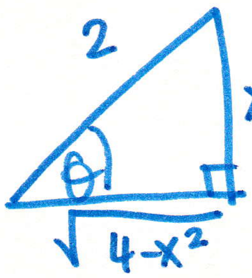
$$\int \frac{1}{\sqrt{4-x^2}} dx = \int \frac{1}{\cancel{2\cos\theta}} (\cancel{2\cos\theta} \, d\theta)$$

$$= \int d\theta$$

$$= \theta + C$$

$$= \arctan\left(\frac{x}{\sqrt{4-x^2}}\right) + C$$

#



$$\tan\theta = \frac{x}{\sqrt{4-x^2}}$$

$$\boxed{\theta = \arctan\left(\frac{x}{\sqrt{4-x^2}}\right)}$$

$$\theta = \arcsin\left(\frac{x}{2}\right)$$

$$\frac{p(x)}{q(x)}$$ พหุนาม / พหุนาม $\Longrightarrow$ ฟังก์ชันตรรกยะ (rational function)

$$\frac{x-1}{x^2+2x+2}$$

$\leftarrow \deg P(x) = 1$

$\leftarrow \deg Q(x) = 2$

Ex $$\int \frac{dx}{x^2+x-2} = \int \frac{1}{(x+2)(x-1)}\,dx$$

partial fraction เศษส่วนย่อย

$$\frac{1}{6} = \frac{1}{2 \cdot 3} = \frac{♥}{2} + \frac{★}{3}$$

♥ = ? ★ = ?

$$\frac{1}{6} = \frac{3♥ + 2★}{6}$$

$$1 = 3♥ + 2★$$

$$♥ = 1, \quad ★ = -1$$

$$\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$$

$\frac{P(x)}{q(x)}$: ดีกรีของตัวเศษ < ดีกรีของตัวส่วนเสมอ

7.5 Integrating rational functions by partial fractions

Recall that a rational function is a function that can be written as a quotient of two polynomials. Assume that $f(x) = \dfrac{P(x)}{Q(x)}$ is a rational function, where $P(x)$ and $Q(x)$ are polynomials. If $\deg P(x) < \deg Q(x)$, then $f(x)$ is said to be *proper*. If $\deg P(x) \geq \deg Q(x)$, then $f(x)$ is said to be *improper* .

We now find the form of partial fraction decomposition of a proper rational function $f(x) = \dfrac{P(x)}{Q(x)}$. Elementary algebra tells us that $Q(x)$ has only two types of irreducible factors which are of degree 1 or degree 2. Therefore, the partial fraction decomposition of $f(x)$ can be determined by using the following rules, **Linear factor and Quadratic factor rules**.

Linear factor rule: For each factor of the form $(ax+b)^m$, the partial fraction decomposition contains the following sum of m partial fractions:
$\dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \ldots + \dfrac{A_m}{(ax+b)^m}$, where $A_i \quad (i = 1, 2, \ldots, m)$ are constants.

Case 1 $Q(x) = (x-a_1)(x-a_2)\cdot\ldots\cdot(x-a_n)$; $a_i \neq a_j$, $i \neq j$

$$\frac{P(x)}{Q(x)} = \frac{A_1}{(x-a_1)} + \frac{A_2}{(x-a_2)} + \frac{A_3}{(x-a_3)} + \ldots + \frac{A_n}{(x-a_n)}$$

หาค่า $A_1, A_2, \ldots, A_n$

Case 2 $Q(x) = (x-a_1)(x-a_2)\cdot\ldots\cdot(x-a_n)$ โดยที่มีบาง a_i ที่ซ้ำกัน

Ex $$\frac{P(x)}{(x-a)^3} = \frac{A_1}{(x-a)} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3}$$

$$\frac{P(x)}{(x-1)(x-2)^3} = \boxed{\frac{A_1}{x-1}} + \boxed{\frac{A_2}{(x-2)} + \frac{A_3}{(x-2)^2} + \frac{A_4}{(x-2)^3}}$$

$$\frac{P(x)}{(x-2)^2(x-4)^3} = \boxed{\frac{A_1}{x-2} + \frac{A_2}{(x-2)^2}} + \boxed{\frac{A_3}{(x-4)} + \frac{A_4}{(x-4)^2} + \frac{A_5}{(x-4)^3}}$$

Case 3 $Q(x)$ เป็นตัวประกอบกำลังสองที่แยก ต.ป.ก. ไม่ได้

Ex $$\frac{P(x)}{(x-a)(x^2+bx+c)} = \frac{A_1}{(x-a)} + \frac{A_2x+B_2}{x^2+bx+c}$$

Case 4 ตัวประกอบกำลังสองซ้ำกัน

Ex $$\frac{P(x)}{(x-2)(x-3)^2(x^2+x+2)^2} = \frac{A_1}{x-2} + \frac{A_2}{(x-3)} + \frac{A_3}{(x-3)^2} + \frac{A_4x+B_4}{x^2+x+2} + \frac{A_5x+B_5}{(x^2+x+2)^2}$$

Example 7.14 Evaluate $\displaystyle\int \frac{dx}{x^2+x-2}$.

$$\int \frac{dx}{(x+2)(x-1)}$$

พิจารณา $\dfrac{1}{x^2+x-2} = \dfrac{A}{x+2} + \dfrac{B}{(x-1)}$

$$\frac{1 \quad P(x)}{Q(x) \quad (x+2)(x-1)} = \frac{A}{(x+2)} + \frac{B}{(x-1)} \quad -(*)$$

คูณ (*) ทั่วสมการด้วย $(x+2)(x-1)$

$$1 = A(x-1) + B(x+2)$$

$$P(x) \quad 1 = Ax - A + Bx + 2B$$

$$0x + 1 = (A+B)x + (-A+2B)$$

เทียบ สปส. $A + B = 0 \quad -(1)$

$$-A + 2B = 1 \quad -(2)$$

(1)+(2) $\quad 3B = 1 \Rightarrow B = \frac{1}{3}$

แทน(1) $\quad A = -\frac{1}{3}$

$$\therefore \frac{1}{(x+2)(x-1)} = -\frac{1}{3}\cdot\frac{1}{(x+2)} + \frac{1}{3}\cdot\frac{1}{(x-1)}$$

$$\therefore \int \frac{1}{(x+2)(x-1)}dx = \int\left(-\frac{1}{3}\cdot\frac{1}{(x+2)} + \frac{1}{3}\cdot\frac{1}{x-1}\right)dx$$

$$= \frac{1}{3}\int\left[\left(-\frac{1}{x+2}\right) + \left(\frac{1}{x-1}\right)\right]dx$$

$$= -\frac{1}{3}\int\frac{1}{x+2}dx + \frac{1}{3}\int\frac{1}{x-1}dx$$

$$= -\frac{1}{3}\ln|x+2| + \frac{1}{3}\ln|x-1| + C \quad \# \quad ✓$$

$$= \ln\left(\frac{|x-1|}{|x+2|}\right)^{\frac{1}{3}} + C \quad ✓$$

Integrating improper rational functions

Example 7.19 Evaluate $\displaystyle\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx.$

The integrand can be expressed as

$$\frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2} = \underbrace{(3x^2+1)}_{R(x)} + \frac{1}{x^2+x-2}\quad \frac{p(x)}{q(x)}$$

and hence

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx = \int (3x^2+1)\,dx + \int \frac{1}{x^2+x-2}\,dx = x^3+x+\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right| + C.$$

$$
\begin{array}{r|l}
 & 3x^2 \qquad\qquad + 1 \\
\hline
x^2+x-2 & 3x^4+3x^3-5x^2+x-1 \\
 & \pm 3x^4 \pm 3x^3 \mp 6x^2 \\
\hline
 & \qquad\qquad x^2+x-1 \\
 & \qquad\qquad x^2+x-2 \\
\hline
 & \qquad\qquad\qquad +1
\end{array}
$$

ตัวตั้ง = ตัวหาร x ผลหาร + เศษ

$$3x^4+3x^3-5x^2+x-1 = (3x^2+1)(x^2+x-2) + \underline{1}$$

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx = \int \left(3x^2+1+\frac{1}{(x^2+x-2)}\right)dx$$

Example 7.15 Evaluate $\int \frac{2x^2-3x+4}{(x+1)(x-2)^2}\,dx$.

$$\frac{2x^2-3x+4}{(x+1)(x-2)^2} = \frac{A_1}{x+1} + \boxed{\frac{A_2}{x-2} + \frac{A_3}{(x-2)^2}}$$ หา A_1, A_2, A_3 —(*)

คูณ (*) ด้วย $(x+1)(x-2)^2$

$$2x^2-3x+4 = A_1(x-2)^2 + A_2(x+1)(x-2) + A_3(x+1)$$

$$2x^2-3x+4 = A_1(x^2-4x+4) + A_2(x^2-x-2) + A_3(x+1)$$

$$2x^2-3x+4 = A_1x^2 - 4A_1x + 4A_1 + A_2x^2 - A_2x - 2A_2 + A_3x + A_3$$

$$2x^2-3x+4 = (A_1+A_2)x^2 + (-4A_1-A_2+A_3)x + (4A_1+A_3-2A_2)$$

$$A_1 + A_2 = 2 \quad —①$$

$$-4A_1 - A_2 + A_3 = -3 \quad —②$$

$$4A_1 - 2A_2 + A_3 = 4 \quad —③$$

$\Rightarrow$ ~~②+③ ; $-3A_2 + 2A_3 = 1$~~ —

โดย ③ − ② ; $8A_1 - A_2 = 7$ —④

โดย ④ + ① ; $9A_1 = 9 \Rightarrow A_1 = 1$

$\therefore A_2 = 1$

$-3 + 2A_3 = 1 \Rightarrow 2A_3 = 4$

$\therefore A_3 = 2$

$$\therefore \frac{2x^2-3x+4}{(x+1)(x-2)^2} = \frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{(x-2)^2}$$

$$\int \frac{2x^2-3x+4}{(x+1)(x-2)^2}\,dx = \int \left(\frac{1}{x+1} + \frac{1}{x-2} + \frac{2}{(x-2)^2}\right)dx$$

$$= \ln|x+1| + \ln|x-2| - \frac{2}{x-2} + C$$

$$= \ln|(x+1)(x-2)| - \frac{2}{x-2} + C$$

หา $2\int \frac{1}{(x-2)^2}\,dx$ ให้ $u = x-2$, $du = dx$

$$= 2\int \frac{1}{u^2}\,du = -2 \cdot \frac{1}{u} + c$$

$$= -\frac{2}{(x-2)} + C$$

> **Quadratic factor rule :** For each factor of the form $(ax^2+bx+c)^m$ with $b^2-4ac<0$, the partial fraction decomposition contains the following sum of m partial fractions:
> $$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\ldots+\frac{A_mx+B_m}{(ax^2+bx+c)^m},$$ where A_i, B_i $(i=1,2,\ldots,m)$ are constants.

Example 7.16 Evaluate $\displaystyle\int \frac{3x^2+x-2}{(x-1)(x^2+1)}\,dx.$

พิจารณา $\dfrac{3x^2+x-2}{(x-1)(x^2+1)} = \dfrac{A}{(x-1)} + \dfrac{Bx+C}{x^2+1}$

$\therefore$ $3x^2+x-2 = A(x^2+1)+(Bx+C)(x-1)$

$3x^2+x-2 = Ax^2+A+Bx^2-Bx+Cx-C$

$3x^2+x-2 = (A+B)x^2+(-B+C)x+(A-C)$

เทียบ สปส.

$A+B = 3$ — (1)

$-B+C = 1$ — (2)

$A-C = -2$ — (3)

(1)+(2); $A+C = 4$ — (4)

(3)+(4); $2A = 2 \Rightarrow A = 1$

$\therefore C = 3,\ B = 2$

$\therefore \dfrac{3x^2+x-2}{(x-1)(x^2+1)} = \dfrac{1}{x-1} + \dfrac{2x+3}{x^2+1}$

$\therefore \displaystyle\int \frac{3x^2+x-2}{(x-1)(x^2+1)}dx = \int\left(\frac{1}{x-1}+\left(\frac{2x+3}{x^2+1}\right)\right)dx$ — (*)

จาก $\displaystyle\int\frac{2x+3}{x^2+1}dx = \boxed{\int\frac{2x}{x^2+1}dx} + \boxed{\int\frac{3}{x^2+1}dx}$

$u = x^2+1$

$du = 2x\,dx$

$\displaystyle\int\frac{2x}{x^2+1} = \int\frac{1}{u}du = \ln|u|+C_1$

$\displaystyle\int\frac{1}{x^2+1}dx = \arctan(x)+C_2$

(*) $= \ln|x-1|+\ln|x^2+1| + 3\arctan x + C$ #

Example 7.17 Evaluate $\displaystyle\int \frac{x+4}{x^2(x^2+4)}\,dx.$

หาค่าคงตัว A, B, C, D

[illegible] $\dfrac{x+4}{x^2(x^2+4)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+4}$ $\left(C = -\frac{1}{4},\ D = -1,\ A = \frac{1}{4},\ B = 1 \right)$

Example 7.18 Evaluate $\displaystyle\int \frac{x^3-4x}{(x^2+1)^2}\,dx.$

หาได้ว่า A,B,C,D

พิจารณา $\dfrac{x^3-4x}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$ $\begin{pmatrix} A=1, & B=0 \\ C=-5, & D=0 \end{pmatrix}$

Example 7.17 Evaluate $\displaystyle\int \frac{x+4}{x^2(x^2+4)}\,dx.$

แยก $\dfrac{x+4}{x^2(x^2+4)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{Cx+D}{x^2+4}$

$\therefore \quad x+4 = Ax(x^2+4) + B(x^2+4) + (Cx+D)x^2$

$x+4 = Ax^3 + 4Ax + Bx^2 + 4B + Cx^3 + Dx^2$

เทียบ สปส.

$A + C = 0 \Rightarrow C = -\frac{1}{4}$

$B + D = 0 \Rightarrow D = -1$

$4A = 1 \Rightarrow A = \frac{1}{4}$

$4B = 4 \Rightarrow B = 1$

$\therefore \displaystyle\int \frac{x^2+4}{x^2(x^2+4)}\,dx = \frac{1}{4}\int \frac{1}{x}\,dx + \int \frac{1}{x^2}\,dx + \int \frac{-\frac{1}{4}x - 1}{x^2+4}\,dx$

$= \frac{1}{4}\ln|x| - \frac{1}{x} - \frac{1}{4}\boxed{\int \frac{x+4}{x^2+4}\,dx}$

$= \frac{1}{4}\ln|x| - \frac{1}{x} - \frac{1}{4}\left(\frac{1}{2}\ln|x^2+4| + 2\arctan\left(\frac{x}{2}\right)\right) + C$

$= \frac{1}{4}\ln|x| - \frac{1}{x} - \frac{1}{8}\ln|x^2+4| - \frac{1}{2}\arctan\left(\frac{x}{2}\right) + C$ #

$\displaystyle\int \frac{x+4}{x^2+4}\,dx$ $\quad u = x^2+4$, $du = 2x\,dx$

$= \frac{1}{2}\displaystyle\int \frac{2x+8}{x^2+4}\,dx$

$= \frac{1}{2}\displaystyle\int \frac{1}{u}\,du + \frac{8}{2}\int \frac{1}{x^2+4}\,dx$

$= \frac{1}{2}\ln|u| + \frac{4}{2}\arctan\left(\frac{x}{2}\right) + C$

$= \frac{1}{2}\ln|x^2+4| + 2\arctan\left(\frac{x}{2}\right) + C$

Example 7.18 Evaluate $\displaystyle\int \frac{x^3-4x}{(x^2+1)^2}\,dx.$

แยก $\dfrac{x^3-4x}{(x^2+1)^2} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{(x^2+1)^2}$

$\therefore \quad x^3-4x = (Ax+B)(x^2+1) + Cx+D$

$x^3-4x = Ax^3 + Ax + Bx^2 + Cx + D$

เทียบ สปส.

$A = 1$

$B = 0$

$A + C = -4 \Rightarrow C = -5$

$D = 0$

$\therefore \displaystyle\int \frac{x^3-4x}{(x^2+1)^2}\,dx = \int \left(\frac{x}{x^2+1} - \frac{5x}{(x^2+1)^2}\right)dx$

$= \boxed{\displaystyle\int \frac{x}{x^2+1}\,dx} - 5\boxed{\displaystyle\int \frac{x}{(x^2+1)^2}\,dx}$

$= \frac{1}{2}\ln|x^2+1| + \frac{5}{2}\left(\frac{1}{x^2+1}\right) + C$ #

$\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u^2}\,du$

$= -\frac{1}{2u}$

$= -\frac{1}{2(x^2+1)}$

$\displaystyle\int \frac{x}{x^2+1}\,dx$

$u = x^2+1$

$du = 2x\,dx$

$\therefore \displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du$

$= \frac{1}{2}\ln|u|$

$= \frac{1}{2}\ln|x^2+1|$