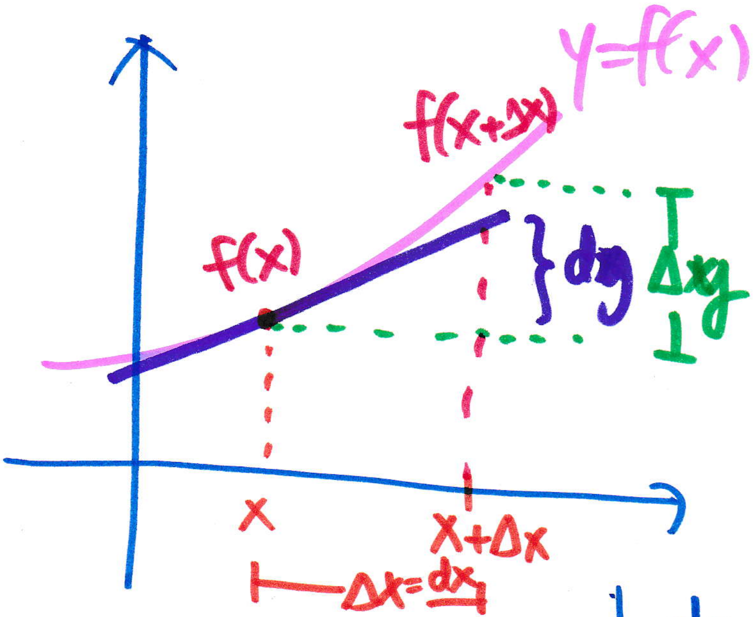
Differential



$$\Delta y \approx dy$$

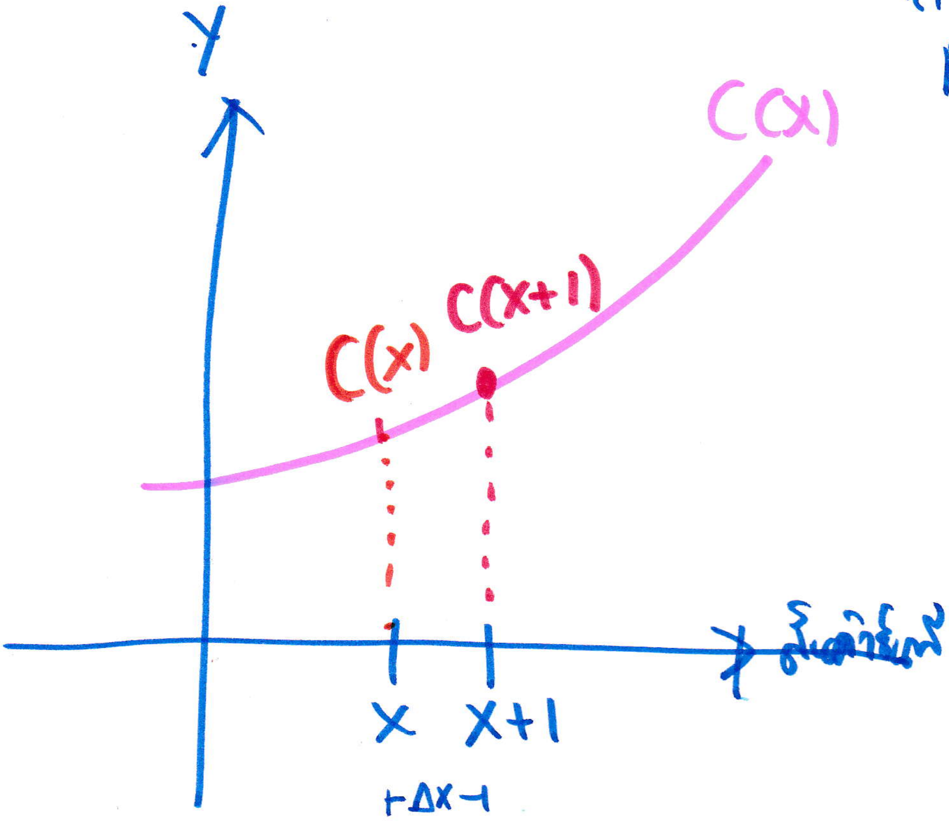
$$dy = f'(x)dx$$

$\therefore$ ประมาณ $\Delta y \approx f'(x)dx$

$$= f'(x)\Delta x$$

- Price-demand function $p(x)$
- Revenue function $R(x) = x \cdot p(x)$
- Cost function $C(x) = a + bx$
- Profit function $P(x) = R(x) - C(x)$

สมมติว่ามีสินค้า x ชิ้น ต้องการดูว่าชิ้นที่ $x+1$ มีการเปลี่ยนแปลง
→ ต้นทุน $C(x)$
→ รายได้ $R(x)$
→ กำไร $P(x)$
เท่าไร



Exact cost of producing the $(x+1)^{st}$ item

$$= C(x+1) - C(x)$$

การวิเคราะห์หน่วยท้ายสุด

5.10 Marginal Analysis

Let $C(x)$ be the total cost of producing x items. We call $C'(x)$ the marginal cost function. $C'(x)$ represents the instantaneous rate of change of total cost with respect to the number of items produced. Similar statements can be made for total revenue functions and total profit functions.

If x is the number of units of a product produced in some time interval, then

total cost	$= C(x)$
marginal cost	$= C'(x)$
total revenue	$= R(x)$
marginal revenue	$= R'(x)$
total profit	$= P(x) = R(x) - C(x)$
marginal profit	$= P'(x) = R'(x) - C'(x)$

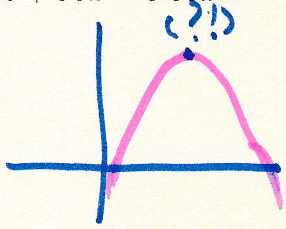
Remark: $C(x)$ represents the total cost of producing x items, not the cost of producing a single item. To find the cost of producing a single item, we use the difference of two successive values of $C(x)$:

Total cost of producing $x+1$ items	$= C(x+1)$
Total cost of producing x items	$= C(x)$
Exact cost of producing the $(x+1)^{st}$ item	$= C(x+1) - C(x)$

Example 5.10.1. A company manufactures fuel tanks for cars. The total weekly cost (in dollars) of producing x tanks is given by $C(x) = 10,000 + 90x - 0.05x^2$.

1. Find the marginal cost function.

$$C'(x) = 90 - 0.1x$$



2. Find the marginal cost at a production level of 500 tanks per week and interpret the results.

จาก $C'(500) = 90 - 0.1(500)$
$= 90 - 50 = 40$

ตอบ ที่การผลิต 500 ตัวต่อสัปดาห์
ต้นทุนการผลิตจะเพิ่มขึ้นด้วยอัตรา 40 dollar/tank

$C(501) - C(500)$ $\qquad C(x) = 10{,}000 + 90x - 0.05x^2$

3. Find the exact cost of producing the 501st item. $C'(x) = 90 - 0.1x$

$C(501) = 10{,}000 + 90(501) + 0.05(501)^2 = 42{,}539.95$ \$

$C(500) = 10{,}000 + 90(500) + 0.05(500)^2 = 42{,}500$ \$

$\boxed{C(501) - C(500)} = 42{,}539.95 - 42{,}500 = 39.95$ \$

$\Delta x = 1 \qquad C'(x) \approx \dfrac{C(x+\Delta x) - C(x)}{\Delta x}$

$C(501) - C(500) \approx C'(500)$

$\therefore C'(500) = 40$ \$

Remark: Increments and differentials will help us understand the relationship between marginal cost and the cost of a single item. If $C(x)$ is any total cost function, then

$$
\begin{aligned}
C'(x) &\approx \frac{C(x+\triangle x) - C(x)}{\triangle x} \\
C'(x) &\approx C(x+1) - C(x). \qquad (\triangle x = 1)
\end{aligned}
$$

It can be seen that the marginal cost $C'(x)$ approximates $C(x+1) - C(x)$, the exact cost of producing the $(x+1)^{\text{st}}$ item.

Marginal cost and exact cost

If $C(x)$ is the total cost of producing x items, then the marginal cost function approximates the exact cost of producing the $(x+1)^{\text{st}}$ items:

$$C'(x) \approx C(x+1) - C(x)$$

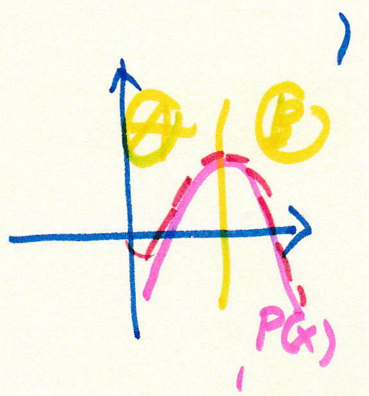
Similar statements can be made for total revenue functions and total profit functions.

Example 5.10.2. The total cost and the total revenue (in Thai Baht) of making and selling x cups of iced coffee in 1 day are given by $\underset{\text{Cost}}{\underline{C(x)}} = 1,000 + 6x - \frac{1}{100}x^2$ and $\underset{\text{Revenue}}{\underline{R(x)}} = 40x - \frac{x^2}{10}$, $\quad 0 \le x \le 450$.

1. Find the marginal cost function

$$C'(x) = 6 - \frac{1}{50}x$$

2. Find the profit function and the marginal profit function.



$$\begin{aligned} P(x) &= R(x) - C(x) \\ &= \left(40x - \frac{x^2}{10}\right) - \left(1000 + 6x - \frac{1}{100}x^2\right) \\ &= -1000 + 34x - \frac{9x^2}{100} \end{aligned}$$

$$P'(x) = 34 - \frac{9}{50}x$$

3. Find the marginal profit at $\underline{x = 200}$. Then explain these quantities.

$$\begin{aligned} P'(200) &= 34 - \frac{9}{50}(200) \\ &= 34 - 36 = -2 \end{aligned}$$

เมื่อผลิตกาแฟเย็นที่ 200 แก้ว จะเห็นว่า กำไรลดลงด้วยอัตรา ๒ บาท/แก้ว

4. Approximate the profit of selling the 101^{st} cup of iced coffee.

$$\begin{aligned} P(101) - P(100) &\approx P'(100) \\ &= 34 - \frac{9}{\cancel{50}_{1}}(\cancel{100}^{2}) \\ &= 34 - 18 = 16 \end{aligned}$$

กาแฟเย็น แก้วที่ 101 กำไร ประมาณ 16 บาท

$$\lim_{x\to 2}\frac{x^2-4}{x-2}\ \left(\frac{0}{0}\right) = \lim_{x\to 2}\frac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}} = 4$$

$$\lim_{x\to 1}\frac{e^x-e}{x-1}\ \left(\frac{0}{0}\right) \longrightarrow \text{L'Hôpital Rule.} \longleftarrow ✓$$

L'Hospital Rule

$$\lim_{x\to\infty}\frac{(x^2+3x+8)}{x^3+1} = \lim_{x\to\infty}\frac{\frac{x^2}{x^3}+\frac{3x}{x^3}+\frac{8}{x^3}}{\frac{x^3}{x^3}+\frac{1}{x^3}}$$

$$= \lim_{x\to\infty}\frac{\cancel{\frac{1}{x}}^{\to 0}+\cancel{\frac{3}{x^2}}^{\to 0}+\cancel{\frac{8}{x^3}}^{\to 0}}{1+\cancel{\frac{1}{x^3}}^{\to 0}}$$

$$= 0 \quad \#$$

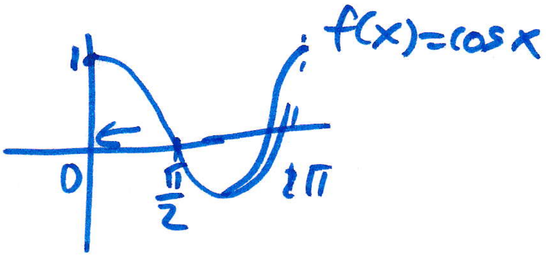
$$\lim_{x\to 0}\frac{x+6}{x+2} \overset{①}{=} \boxed{\lim_{x\to 0}\frac{1}{1} = 1} \quad ✗$$

$\overset{②}{=}$ $\frac{6}{3} = 2$ ✓

จะเห็นว่า $\lim_{x\to 0} x+6 = 6$, $\lim_{x\to 0} x+2 = 2$ $\Big\}\ \neq 0$

$$\lim_{x\to 0^+}\frac{2+x}{1-\cos x} \overset{\text{L'Hopital}\ ①}{=} \boxed{\lim_{x\to 0^+}\frac{1}{1+\sin x} = 1} \quad ✗$$

$\overset{②}{=}$ $+\infty$



รูปแบบไม่กำหนด

5.11 Indeterminate Forms $\frac{0}{0}, \frac{\infty}{\infty}, 0\cdot\infty, \infty-\infty, 0^0, 1^\infty, \infty^0$

L'Hopital s Rule กฎของโลปิตาล.

Let f and g be differentiable on an open interval (a, b) containing c. Suppose that $\lim_{x\to c} f(x) = 0$ and $\lim_{x\to c} g(x) = 0$, or that $\lim_{x\to c} f(x) = \pm\infty$ and $\lim_{x\to c} g(x) = \pm\infty$. (In other words, $\lim_{x\to c} \frac{f(x)}{g(x)}$ is in indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$.) Then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)},$$

if $\lim_{x\to c} \frac{f'(x)}{g'(x)}$ exists, is ∞ or is $-\infty$.

Note that L Hopital s Rule remains valid if the symbol $x \to c$ is replaced everywhere it occurs with one of the following symbols:

$$x \to c^+, x \to c^-, x \to \infty \quad \text{and} \quad x \to -\infty$$

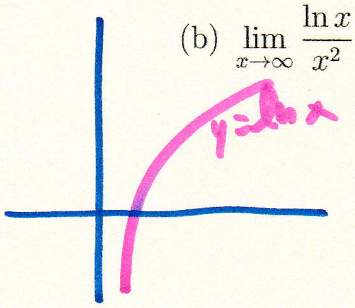
Example 5.11.1. Find the following limits (if exists).

(a) $\lim_{x\to 1} \frac{e^x - e}{x - 1}$

เนื่องจาก $\lim_{x\to 1} e^x - e = 0$, $\lim_{x\to 1} x - 1 = 0$

จะได้ $\lim_{x\to 1} \frac{e^x - e}{x-1} = \lim_{x\to 1} \frac{\frac{d}{dx}(e^x - e)}{\frac{d}{dx}(x-1)} = \lim_{x\to 1} \frac{e^x}{1} = e$ #

(b) $\lim_{x\to\infty} \frac{\ln x}{x^2}$



จาก $\lim_{x\to\infty} \ln x = +\infty$, $\lim_{x\to\infty} x^2 = +\infty$

จะได้ $\lim_{x\to\infty} \frac{\ln x}{x^2} = \lim_{x\to\infty} \frac{\frac{1}{x}}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = 0$ #

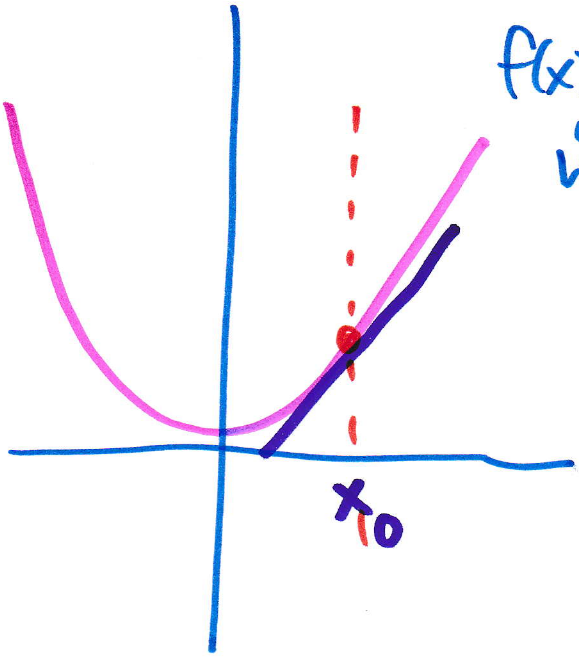
(c) $\lim_{x\to 2} \frac{x^2 - 4}{x - 2}$

เนื่องจาก $\lim_{x\to 2} x^2 - 4 = 0$, $\lim_{x\to 2} x - 2 = 0$

จะได้ $\lim_{x\to 2} \frac{x^2 - 4}{x - 2} = \lim_{x\to 2} \frac{2x}{1} = 2\cdot 2 = 4$ #

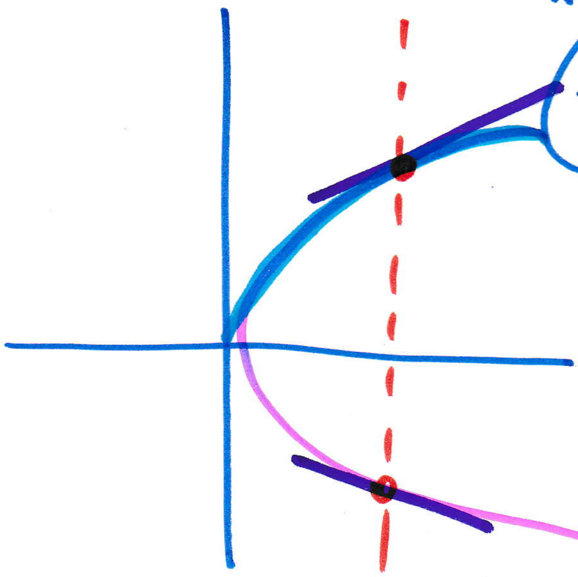
$f(x) = x^2$

ฟังก์ชันชัดแจ้ง (explicit function)



ฟังก์ชันแฝง (implicit function)

ฟังก์ชันโดยปริยาย



$y^2 = x$

$\Rightarrow y = \sqrt{x}$

$\Rightarrow y = -\sqrt{x}$

(d) $\lim_{x \to \infty} \frac{x^2 + 3x + 8}{x^3 + 1} \overset{\text{L'Hôpital}}{=} \lim_{x \to \infty} \frac{(2x+3)}{3x^2}$

$\overset{\text{L'Hôpital}}{=} \lim_{x \to \infty} \frac{2}{6x} = 0$ #

(e) $\lim_{x \to \infty} \frac{x}{e^x} \overset{\text{L'Hôpital}}{=} \lim_{x \to \infty} \frac{1}{e^x} = 0$ #

5.12 Implicit Differentiation

Consider the following equations involving two variables x and y.

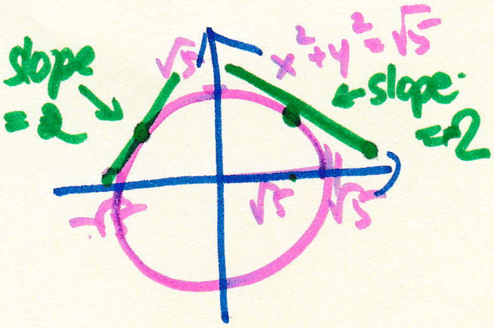
$$x^2y^2 + xy = x + y + 5$$
$$x^2 + y^2 = 25xy$$

We say that above equations define functions y **implicitly as a function of** x.

> **Implicit Differentiation**
>
> To find the derivative y implicitly, we differentiate both sides of the equation with respect to x and consider y as a function of x.

Example 5.12.1. Given $x^2 + y^2 = 5$.

1. Find $\frac{dy}{dx}$ at $(2, 1)$ and $(-2, 1)$.



① Diff ทั้งสองข้าง $\quad \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(5)$

② หาผล Diff $\quad 2x + 2y\boxed{\frac{dy}{dx}} = 0$

③ แก้สมการหา $\frac{dy}{dx}$ $\quad 2y\frac{dy}{dx} = -2x$

$\therefore \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$

$\therefore \left.\frac{dy}{dx}\right|_{(2,1)} = -\frac{2}{1} = -2, \quad \left.\frac{dy}{dx}\right|_{(-2,1)} = -\frac{(-2)}{1} = 2$

2. Find the equations of the tangent lines to the graph of the equation at the points $(2,1)$.

$y - y_1 = m(x - x_1)$

$\therefore\ y - 1 = (-2)(x-2)$

at $(-2,1)$ $\quad y - 1 = (2)(x+2)$

Example 5.12.2. Use implicit differentiation to find $\frac{dy}{dx}$ for

1. $x^3y + \sin y = 11$

$\frac{d}{dx}(x^3y + \sin y) = \frac{d}{dx}(11)$

$\left(x^3\frac{dy}{dx} + y\frac{d(x^3)}{dx}\right) + \frac{d(\sin y)}{dx} = 0$

$\left(x^3\frac{dy}{dx} + 3x^2y\right) + \cos y\frac{dy}{dx} = 0$

$x^3\frac{dy}{dx} + \cos y\frac{dy}{dx} = -3x^2y$

$(x^3 + \cos y)\frac{dy}{dx} = -3x^2y$

$\boxed{\therefore \frac{dy}{dx} = \frac{-3x^2y}{x^3+\cos y}}$ #

2. $e^{xy} - 2x = y + 1$

$\frac{d}{dx}(e^{xy} - 2x) = \frac{d}{dx}(y+1)$

$e^{xy}\frac{d}{dx}(xy) - 2 = \frac{dy}{dx}$

$e^{xy}\left[x\frac{dy}{dx} + y\frac{dx}{dx}\right] - 2 = \frac{dy}{dx}$

$e^{xy}\left(x\frac{dy}{dx} + y\right] - 2 = \frac{dy}{dx}$

$xe^{xy}\frac{dy}{dx} - \frac{dy}{dx} = \cancel{2e^{xy}}y$

$(xe^{xy} - 1)\frac{dy}{dx} = 2 - e^{xy}y$

$\boxed{\therefore \frac{dy}{dx} = \frac{2-e^{xy}y}{xe^{xy}-1}}$ #