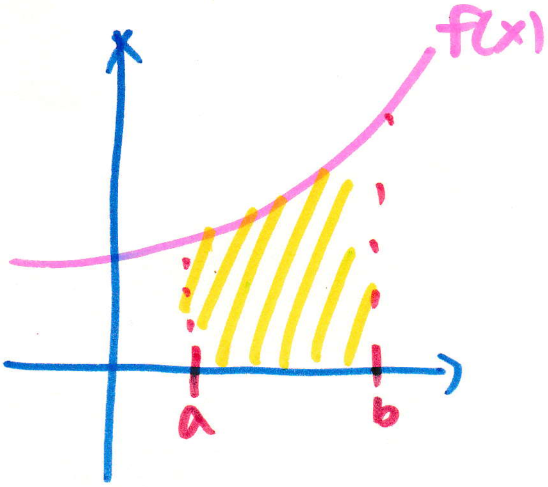
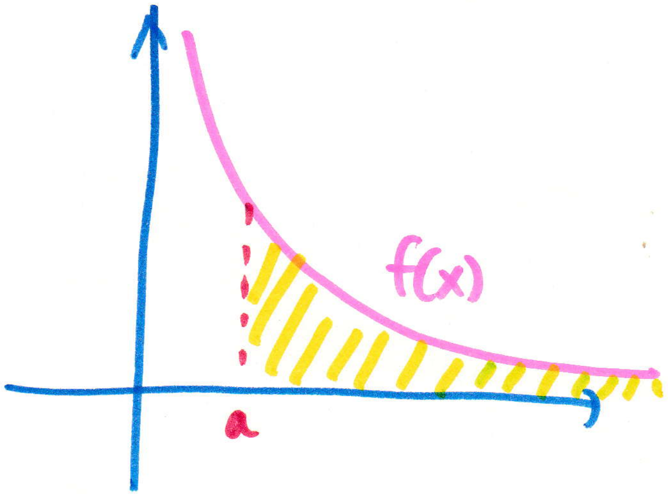
$$\int_a^b f(x)\,dx$$



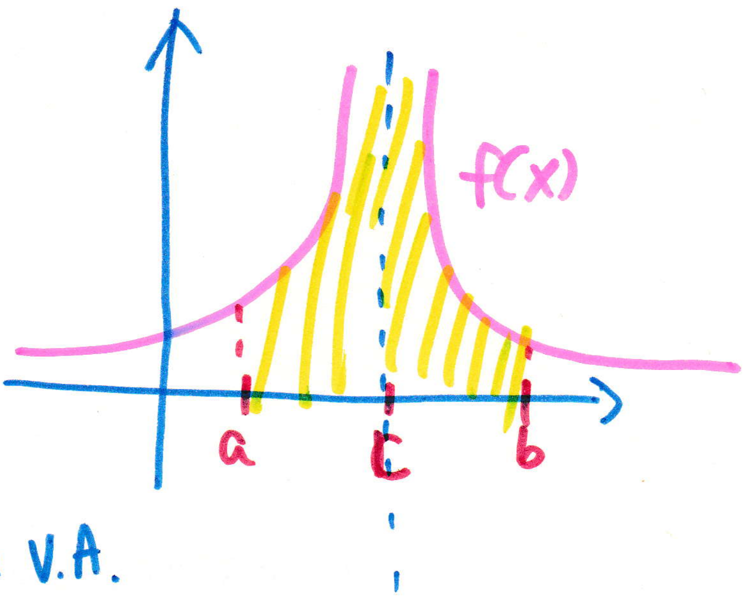
$$\int_a^{+\infty} f(x)\,dx$$



$$\int_a^b f(x)\,dx$$

แต่ $f(x)$ ไม่ต่อเนื่องที่ $x = c$

โดยที่ $x=c$ เป็น V.A.



Ex $\displaystyle\int_0^{\pi} \sec^2 x\,dx = \tan x \Big|_0^{\pi}$

$= \tan(\pi) - \tan(0) = 0$

$f(x)$ discont. ที่ $x = \frac{\pi}{2}$

Q $f(x) = \sec^2 x$ cont ทุกจุดใน $[0, \pi]$: $\sec^2 x = \frac{1}{\cos^2 x}$

7.6 Improper Integrals ปริพันธ์ไม่ตรงแบบ

It is assumed in the definition of the definite integral $\int_a^b f(x)\,dx$ that $[a,b]$ is a finite interval and that the limit that defines the integral exists; that is, the function f is integrable.

Our main objective is to extend the concept of definite integrals for infinite intervals of integration and for integrands with vertical asymptotes within the interval of integration. If a function f has a vertical asymptote, then f is said to have an *infinite discontinuity* .

An integral over an infinite interval of integration or an integral with an infinite discontinuity will be called an *improper integral* .

There are three types of improper integrals:

1. Improper integrals with infinite intervals of integration. (ขอบเขตเป็น ∞)
2. Improper integrals with infinite discontinuities in the interval of integration. (มีบางจุดที่ไม่ต่อเนื่องในช่วง)
3. Improper integrals with infinite discontinuities over infinite intervals of integration. (1+2)

Example 7.20 Determine if each of the following integrals is improper. If so, specify its type.

1. $\displaystyle\int_0^{+\infty} \frac{1}{1-x^2}\,dx$
2. $\displaystyle\int_{-\infty}^{+\infty} x^3\,dx$
3. $\displaystyle\int_0^{\pi} \sec^2\theta\,d\theta$
4. $\displaystyle\int_{-3}^{3} \frac{x}{x^2+x+1}\,dx$

① $\displaystyle\int_0^{+\infty} \frac{1}{1-x^2}\,dx$

(1) ขอบเขตเป็น $+\infty$ ✓

(2) $f(x) = \frac{1}{1-x^2}$ discontinuous ที่ไหนบ้าง. ✓

$x = 1, -1$

ช่วงที่อินทิเกรต $[0, +\infty)$

$\therefore$ เป็น improper integral แบบที่ 3 #

② $\displaystyle\int_{-\infty}^{+\infty} x^3\,dx$

(1) ขอบเขตเป็น $+\infty$ ✓

(2) $f(x) = x^3$ cont บน $(-\infty, \infty)$ $\boxed{\text{NO 2}}$ ×

$\therefore$ เป็น improper integral แบบที่ 1 #

③ $\displaystyle\int_0^{\pi} \sec^2\theta\,d\theta$

(1) ขอบเขตไม่เป็น $\pm\infty$ $\boxed{\text{No 1}}$ ×

(2) มี $x = \frac{\pi}{2}$ ที่ทำให้ $f(x) = \sec^2 x$ ไม่ต่อเนื่องใน $[0, \pi]$

$\therefore$ เป็น improper integral แบบที่ 2 #

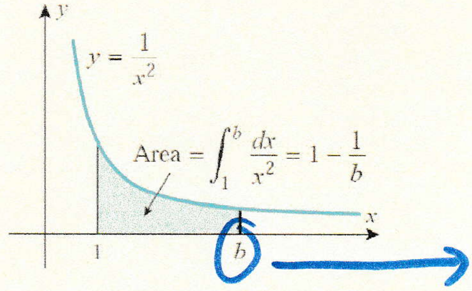
④ $\displaystyle\int_{-3}^{3} \frac{x}{x^2+x+1}\,dx = \int_{-3}^{3} \frac{x}{\left(x^2+2(x)\left(\frac{1}{2}\right)+\left(\frac{1}{4}\right)\right)-\frac{1}{4}+1}\,dx = \int_{-3}^{3} \frac{x}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}\,dx$

$\therefore$ ไม่เป็น improper integral. #

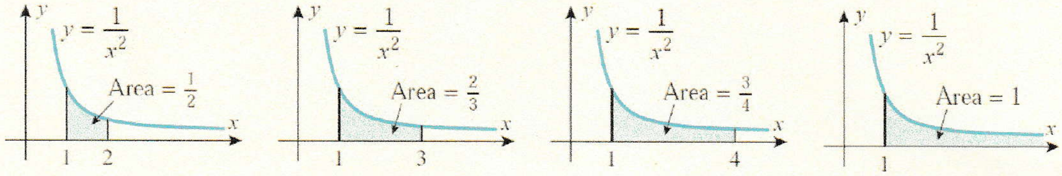
Type 1

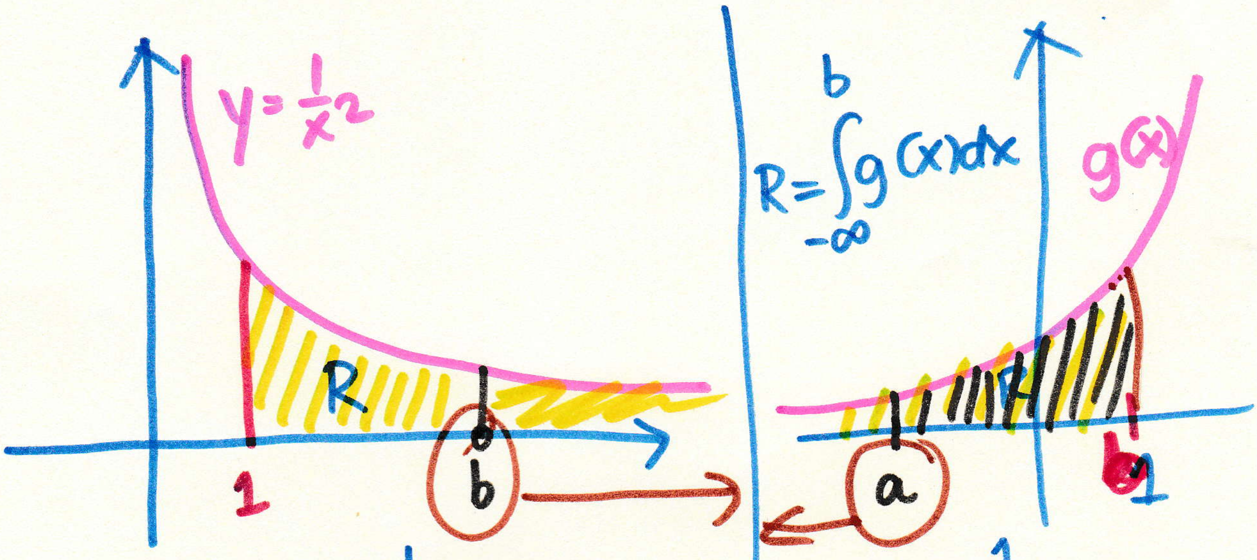
7.6.1 Integrals over infinite intervals :

Suppose we are interested in the area A of the region that lies below the curve $y = 1/x^2$ and above the interval $[1, +\infty)$ on the x-axis. Let us begin by calculating the portion of the area that lies above a finite interval $[1, b]$, where $b > 1$ is arbitrary. That area is $\int_1^b \frac{dx}{x} = 1 - \frac{1}{b}$



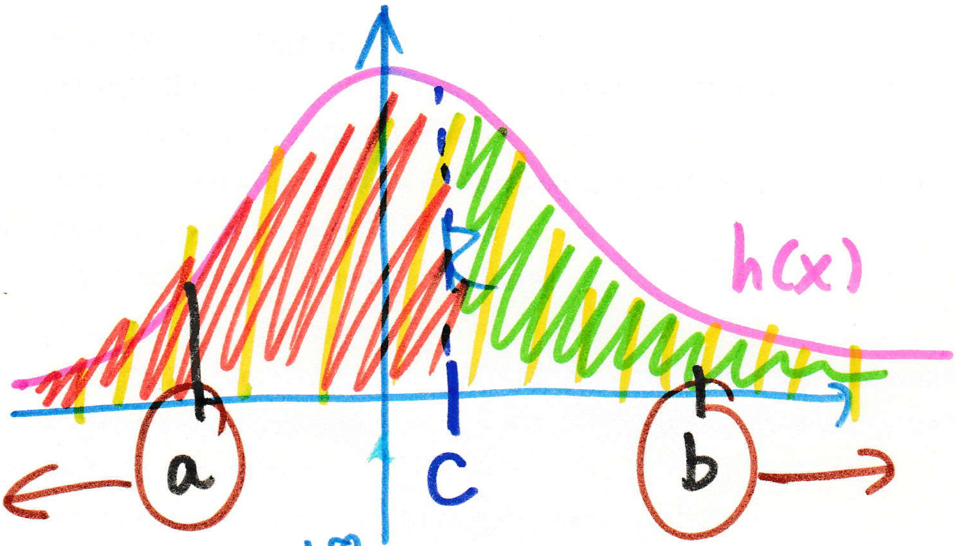
If we now allow b to increase so that $b \to +\infty$, then the portion of the area over the interval $[1, b]$ will begin to fill out the area over the entire interval $[1, +\infty)$, and hence we can reasonably define the area A under $y = 1/x^2$ over the interval $[1, +\infty)$ to be $A = \int_1^\infty \frac{dx}{x} = \lim_{b\to+\infty} \int_1^b \frac{dx}{x} = \lim_{b\to+\infty}(1 - \frac{1}{b}) = 1$





$$\int_1^\infty \frac{1}{x^2}dx = R = \lim_{b\to+\infty} \underbrace{\int_1^b \frac{1}{x^2}dx}$$

$$R = \lim_{a\to-\infty} \int_a^1 g(x)dx$$



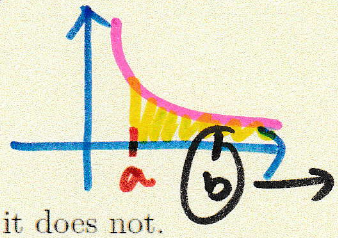
$$R = \int_{-\infty}^{+\infty} h(x)\,dx$$

$$R = \boxed{\int_{-\infty}^{c} h(x)\,dx} + \boxed{\int_{c}^{+\infty} h(x)\,dx}$$

$$R = \lim_{a \to -\infty} \int_{a}^{c} h(x)\,dx + \lim_{b \to +\infty} \int_{c}^{b} h(x)\,dx$$

Definition The improper integral of f over the interval $[a, +\infty)$ is defined to be

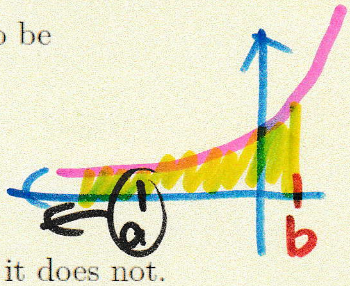
$$\int_a^{\infty} f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx,$$



The integral is said to converge if the limit exists and diverge if it does not.

The improper integral of f over the interval $(-\infty, b]$ is defined to be

$$\int_{-\infty}^{b} f(x)dx = \lim_{a\to-\infty} \int_a^b f(x)dx,$$

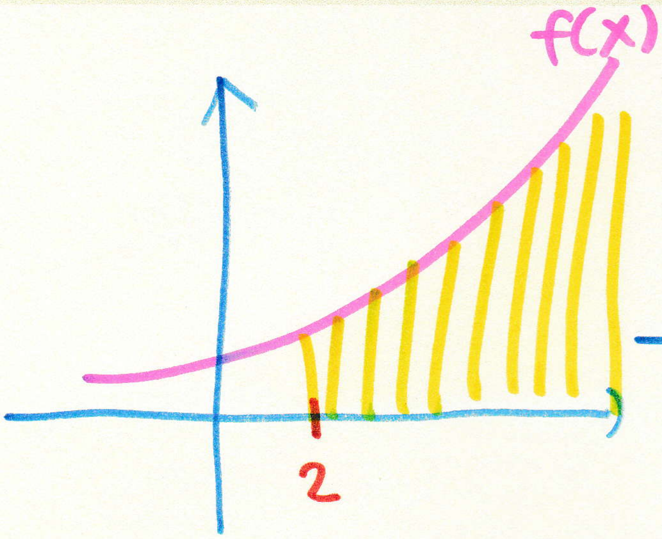


The integral is said to converge if the limit exists and diverge if it does not.

The improper integral of f over the interval $(-\infty, +\infty)$ is defined as

$$\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_c^{\infty} f(x)dx$$

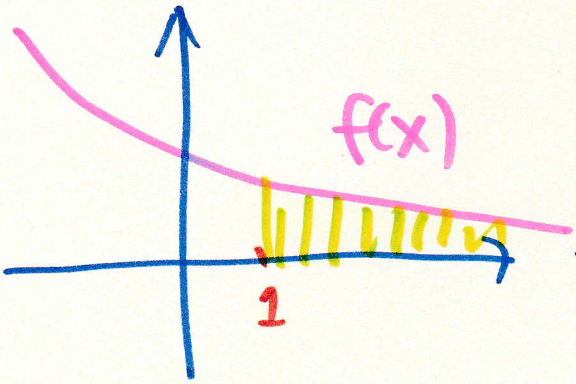
where c is any real number. The improper integral is said to converge if both terms converge and diverge if either term diverges.



$\int_2^{+\infty} f(x)dx$

→ integral diverge

ลู่ออก

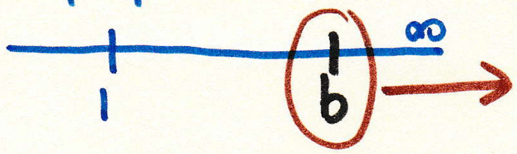


$\int_1^{+\infty} f(x)dx = 5$

→ integral converge

ลู่เข้า

Example 7.21 Compute $\int_1^{+\infty} \frac{1}{x^3}\, dx$. $\longrightarrow$ เป็น improper integral แบบ 1



$$\int_1^{+\infty} \frac{1}{x^3} dx = \lim_{b\to+\infty} \int_1^b \frac{1}{x^3} dx$$

$$= \lim_{b\to+\infty} \left[-\frac{1}{2}\cdot x^{-2}\right]_1^b$$

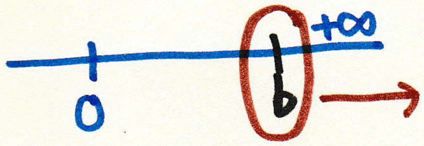
$$= \lim_{b\to+\infty} \left[-\frac{1}{2}\cdot\frac{1}{b^2} + \frac{1}{2}\cdot\frac{1}{(1)^2}\right]$$

$$= \lim_{b\to+\infty} \left[-\frac{1}{2b^2} + \frac{1}{2}\right]$$

$$= \frac{1}{2}$$

$\therefore \int_1^{+\infty} \frac{1}{x^3} dx$ converges to $\frac{1}{2}$ #

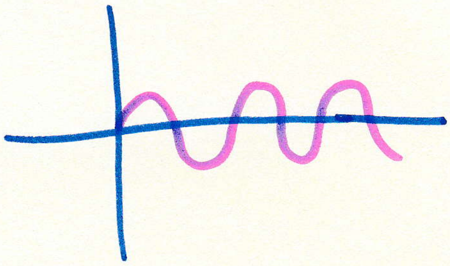
Example 7.22 Compute $\int_0^{+\infty} \cos x\, dx$.



$$\int_0^{+\infty} \cos x\, dx = \lim_{b\to+\infty} \int_0^b \cos x\, dx$$

$$= \lim_{b\to+\infty} \left[+\sin x\right]_0^b$$

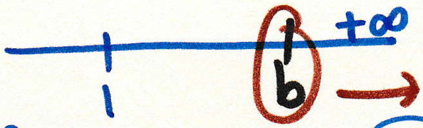
$$= \lim_{b\to+\infty} \left[\sin b - \cancel{\sin 0}^{\,0}\right]$$



เนื่องจาก $\lim_{b\to+\infty} \sin b$ ไม่มีค่าเจาะจง

$\therefore \int_0^{+\infty} \cos x\, dx$ diverges.

Example 7.23 Compute $\int_1^\infty \frac{\ln x}{x}dx$.

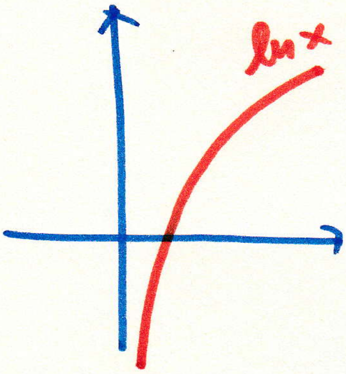


$$\int_1^{+\infty} \frac{\ln x}{x}dx = \lim_{b\to+\infty}\int_1^b \frac{\ln x}{x}dx$$

$$= \lim_{b\to+\infty}\left[\frac{(\ln x)^2}{2}\right]_1^b$$

$$= \lim_{b\to+\infty}\frac{(\ln b)^2}{2} - \cancel{\frac{(\ln 1)^2}{2}}$$

$$= +\infty$$



$\therefore \int_1^{+\infty}\frac{\ln x}{x}dx$ diverges.

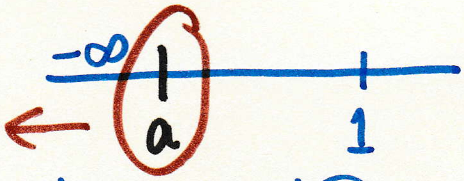
no

$\int \frac{\ln x}{x}dx$

$u = \ln x$; $du = \frac{1}{x}dx$

$$\int \frac{\ln x}{x}dx = \int u\,du = \frac{u^2}{2} + C = \frac{(\ln x)^2}{2} + C$$

Example 7.24 Compute $\int_{-\infty}^1 \frac{1}{3-2x}dx$.



$$\int_{-\infty}^1 \frac{1}{3-2x}dx = \lim_{a\to-\infty}\int_a^1 \frac{1}{3-2x}dx$$

$$= \lim_{a\to-\infty}\left[-\frac{1}{2}\ln|3-2x|\right]_a^1$$

$$= \lim_{a\to-\infty}\left[\cancel{-\frac{1}{2}\ln|1|}^{\,0} + \frac{1}{2}\ln|3-2a|\right]$$

$$= +\infty$$

$\therefore \int_{-\infty}^1 \frac{1}{3-2x}dx$ diverges.

no

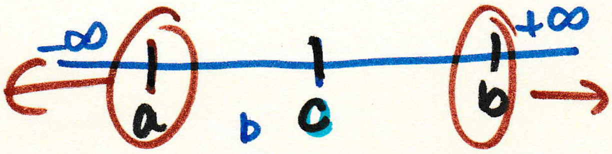
$\int \frac{1}{3-2x}dx$

$u = 3-2x$ $\quad du = -2dx$

$dx = -\frac{1}{2}du$

$$\int \frac{1}{3-2x}dx = -\frac{1}{2}\int\frac{1}{u}du = -\frac{1}{2}\ln|u| + C = -\frac{1}{2}\ln|3-2x| + C$$

Example 7.25 Compute $\int_{-\infty}^{\infty} \frac{x}{x^2+1}dx$.



$$\int_{-\infty}^{\infty} \frac{x}{x^2+1}dx = \lim_{a\to-\infty}\int_a^0 \frac{x}{x^2+1}dx + \lim_{b\to+\infty}\int_0^b \frac{x}{x^2+1}dx$$

$$= \boxed{\lim_{a\to-\infty}\left[\frac{1}{2}\ln|x^2+1|\right]_a^0} + \boxed{\lim_{b\to+\infty}\left[\frac{1}{2}\ln|x^2+1|\right]_0^b}$$

$$= \boxed{\lim_{a\to-\infty}\left[\cancel{\frac{1}{2}\ln|1|}^{\,0} - \frac{1}{2}\ln|a^2+1|\right]}$$

$$+ \boxed{\lim_{b\to+\infty}\left[\frac{1}{2}\ln|b^2+1| - \cancel{\frac{1}{2}\ln|1|}^{\,0}\right]}$$

$$= \boxed{\left(\lim_{a\to-\infty} -\frac{1}{2}\ln|a^2+1|\right)} + \boxed{\lim_{b\to+\infty}\frac{1}{2}\ln|b^2+1|}$$

หา

$$\int \frac{x}{x^2+1}dx$$

$u = x^2+1$; $du = 2x\,dx$

$$\int \frac{x}{x^2+1}dx = \frac{1}{2}\int \frac{1}{u}du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln|x^2+1| + C$$

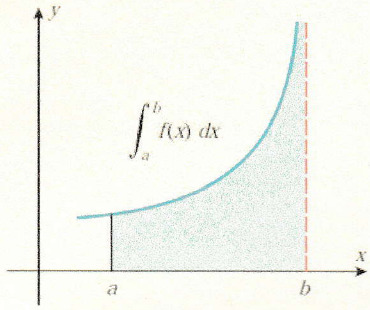
เนื่องจาก $\lim_{a\to-\infty} -\frac{1}{2}\ln|a^2+1|$ diverges และ

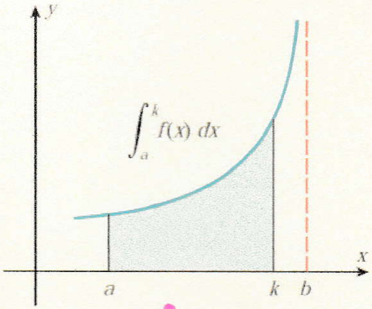
$\lim_{b\to+\infty}\frac{1}{2}\ln|b^2+1|$ diverges

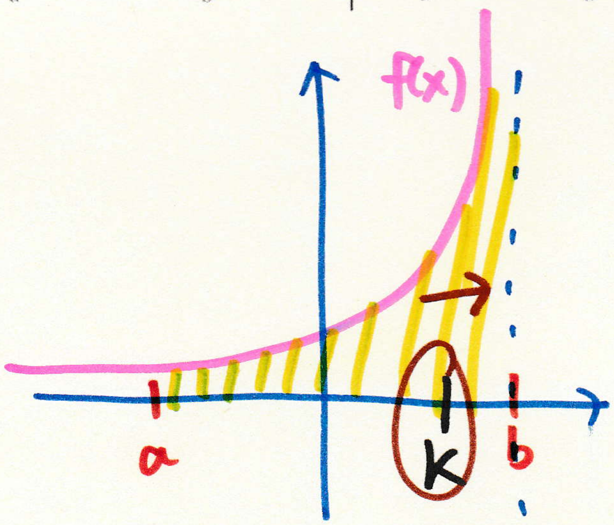
$\therefore \int_{-\infty}^{+\infty}\frac{x}{x^2+1}dx$ diverges. #

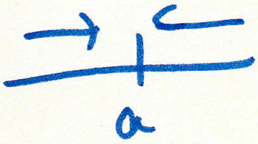
7.6.2 Integrals whose integrands have infinite discontinuities:

Let us consider the case where f is nonnegative on $[a, b]$, so we can interpret the improper integral $\int_a^b f(x)\,dx$ as the area of the region. The problem of finding the area of this region is complicated by the fact that it extends indefinitely in the positive y-direction. However, instead of trying to find the entire area at once, we can proceed indirectly by calculating the portion of the area over the interval $[a, k]$, where $a \le k < b$, and then letting k approach b to fill out the area of the entire region.



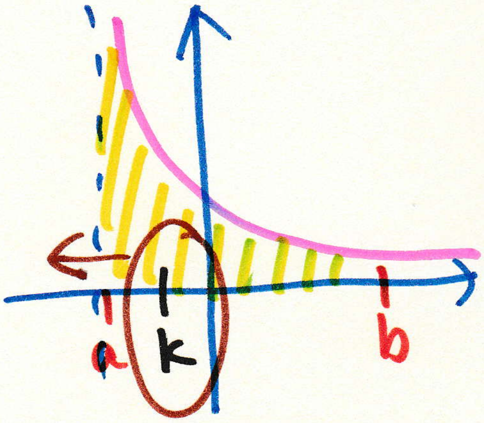






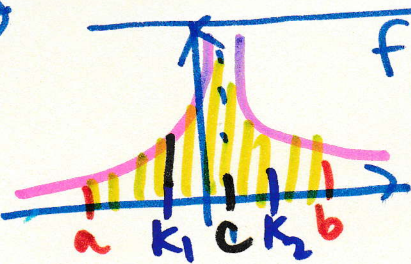
f ไม่ต่อเนื่องที่ $x=b$ แต่ $x=b$ เป็น V.A.

$$\int_a^b f(x)dx = \lim_{k\to b^-}\int_a^k f(x)dx$$



f ไม่ต่อเนื่องที่ $x=a$ แต่ $x=a$ เป็น V.A.

$$\int_a^b f(x)dx = \lim_{k\to a^+}\int_k^b f(x)dx$$

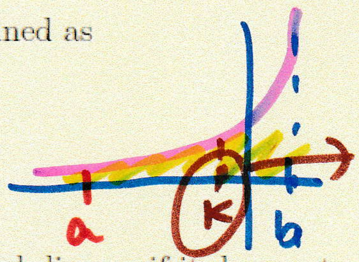


f ไม่ต่อเนื่องที่ $x=c$, $x=c$ เป็น V.A.

$$\int_a^b f(x)dx = \lim_{k_1\to c^-}\int_a^{k_1} f(x)dx + \lim_{k_2\to c^+}\int_{k_2}^b f(x)dx$$

Definition If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at b, then the improper integral of f over the interval $[a, b]$ is defined as

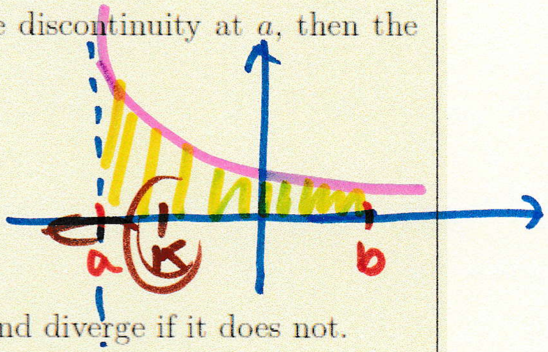
$$\int_a^b f(x)dx = \lim_{k \to b^-} \int_a^k f(x)dx,$$



The integral is said to converge if the indicated limit exists and diverge if it does not.

If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a, then the improper integral of f over the interval $[a, b]$ is defined as

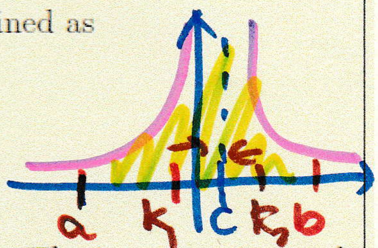
$$\int_a^b f(x)dx = \lim_{k \to a^+} \int_k^b f(x)dx,$$



The integral is said to converge if the indicated limit exists and diverge if it does not.

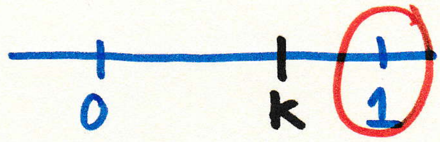
If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a point c in (a, b), then the improper integral of f over the interval $[a, b]$ is defined as

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx,$$



where the two integrals on the right side are themselves improper. The improper integral on the left side is said to converge if both terms on the right side converge and diverge if either term on the right side diverges.

Example 7.26 Compute $\int\limits_0^1 \frac{1}{1-x}\, dx$. ไม่นิยามที่ $x = 1$



$$\int_0^1 \frac{1}{1-x}\, dx = \lim_{k \to 1^-} \int_0^k \frac{1}{1-x}\, dx$$

Example 7.27 Compute $\int\limits_0^3 \frac{dx}{(x-1)^{2/3}}$. ไม่นิยามที่ $x = 1$

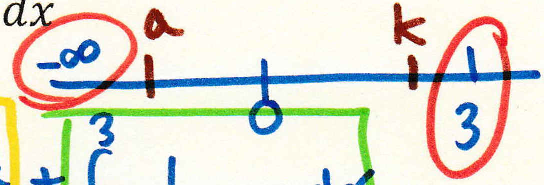
$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \boxed{\lim_{k_1 \to 1^-} \int_0^{k_1} \frac{dx}{(x-1)^{2/3}}} + \lim_{k_2 \to 1^+} \int_{k_2}^3 \frac{dx}{(x-1)^{2/3}}$$

Improper Integral: Third type

Additional Example $\int_{-\infty}^{3} \frac{1}{(x-3)(x-4)}\,dx$

$x = 3, 4$ ไม่ต่อเนื่อง



$$\int_{-\infty}^{3} \frac{1}{(x-3)(x-4)}\,dx = \boxed{\int_{-\infty}^{0} \frac{1}{(x-3)(x-4)}\,dx} + \boxed{\int_{0}^{3} \frac{1}{(x-3)(x-4)}\,dx}$$

$$= \lim_{a \to -\infty} \int_{a}^{0} \frac{1}{(x-3)(x-4)}\,dx + \lim_{k \to 3^-} \int_{0}^{k} \frac{1}{(x-3)(x-4)}\,dx$$

$\vdots$