3. จงหา $\frac{dy}{dx}$ ของฟังก์ชันแฝง $x^4y - 5^y = \sin y$

$$\frac{d}{dx}(x^4y-5^y)=\frac{d}{dx}(\sin y)$$

$$\left[x^2\frac{dy}{dx}+y\frac{d}{dx}(x^4)\right]-\frac{d}{dx}(5^y)=\frac{d}{dx}(\sin y)$$

$$\left(x^2\frac{dy}{dx}+4x^3y\right)-5^y\ln 5\frac{dy}{dx}=\cos y\frac{dy}{dx}$$

$$x^2\frac{dy}{dx}-5^y\ln 5\frac{dy}{dx}-\cos y\frac{dy}{dx}=-4x^3y$$

$$(x^2-5^y\ln 5-\cos y)\frac{dy}{dx}=-4x^3y$$

$$\frac{dy}{dx}=-\frac{4x^3y}{x^2-5^y\ln y-\cos y} \quad \#$$

4. จงหาอนุพันธ์ของฟังก์ชัน $y = \dfrac{2^{4-x}(x^6+8)^{10}}{\sqrt[5]{x^3-1}}$ โดยใช้ลอการิทึมช่วย

## 5.13 Logarithmic Differentiation

**Example 5.13.1.** Use properties of exponential and logarithmic function to find $y'(x)$ for

$\ln(a\cdot b) = \ln(a) + \ln(b)$

1. $y(x) = 5e^{2x^3}(x^2+x)^{3/4}$

Take ln ทั้งสองข้าง.

$$\ln(y(x)) = \ln\left(5e^{2x^3}(x^2+x)^{3/4}\right)$$

$$\ln(y(x)) = \ln 5 + \ln e^{2x^3} + \ln(x^2+x)^{3/4}$$

$$\ln(y(x)) = \ln 5 + 2x^3\ln e + \frac{3}{4}\ln(x^2+x)$$

Diff ทั้งสองข้าง

$$\frac{d}{dx}(\ln(y(x))) = \frac{d}{dx}\left(\ln 5 + 2x^3 + \frac{3}{4}\ln(x^2+x)\right)$$

$$\frac{1}{y(x)}\frac{dy}{dx} = 0 + 6x^2 + \frac{3}{4}\frac{1}{x^2+x}\frac{d}{dx}(x^2+x)$$

$$\frac{1}{y(x)}\frac{dy}{dx} = 6x^2 + \frac{3}{4}\frac{(2x+1)}{x^2+x}$$

$$\frac{dy}{dx} = y(x)\left[6x^2 + \frac{3}{4}\cdot\frac{(2x+1)}{(x^2+x)}\right]$$

$$\frac{dy}{dx} = 5e^{2x^3}(x^2+x)^{3/4}\left[6x^2 + \frac{3}{4}\frac{(2x+1)}{(x^2+x)}\right] \#$$

2. $y(x) = \dfrac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}$

Take ln:

~~$\ln(x^2) + \ln(7x-14)^{1/3}$~~

~~$\ln($~~

$$\ln(y(x)) = \ln\left(\frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\right) \qquad \sqrt[3]{7x-14} = (7x-14)^{1/3}$$

$$\ln(y(x)) = \ln x^2 + \ln(7x-14)^{1/3} - \ln(1+x^2)^4$$

$$\ln(y(x)) = 2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)$$

Diff ทั้งสองข้าง

$$\frac{d}{dx}(\ln(y(x))) = \frac{d}{dx}\left(2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)\right)$$

$$\frac{1}{y(x)}\frac{dy}{dx} = 2\cdot\frac{1}{x} + \frac{1}{3}\frac{1}{7x-14}\frac{d}{dx}(7x-14) - 4\cdot\frac{1}{1+x^2}\frac{d}{dx}(1+x^2)$$

$$\frac{1}{y(x)}\frac{dy}{dx} = \frac{2}{x} + \frac{7}{3}\cdot\frac{1}{7x-14} - \frac{8x}{1+x^2}$$

$$\frac{dy}{dx} = y(x)\left[\frac{2}{x} + \frac{7}{3}\cdot\frac{1}{7x-14} - \frac{8x}{1+x^2}\right]$$

$$\frac{dy}{dx} = \left(\frac{x^2\sqrt[3]{7x-14}}{(1+x^2)^4}\right)\left[\frac{2}{x} + \frac{7}{3}\cdot\frac{1}{7x-14} - \frac{8x}{1+x^2}\right] \#$$

$\underline{\text{Ex}}$ $$f(x) = \frac{5x^{100}\left(\sqrt[10]{x^{200}} + 4\right)}{\cos(x^2+1)\cdot\sin(x)}$$

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

$$\boxed{\begin{aligned} \ln(a\cdot b) &= \ln a + \ln b \\ \ln\left(\frac{a}{b}\right) &= \ln a - \ln b \\ \ln a^n &= n\ln a. \end{aligned}}$$

$$\left.\begin{aligned} \ln(a+b) &\neq \ln a + \ln b \\ \ln(a-b) &\neq \ln a - b \\ \ln a^n &= n \ln a \\ (\ln a)^n &\neq n\ln a \end{aligned}\right\} \text{คำเตือน อย่าทำ!}$$

$$\ln^2 a$$

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$$\ln(a\cdot b) \neq \ln a \cdot \ln b$$

$$\ln\left(\frac{a}{b}\right) \neq \frac{\ln a}{\ln b}$$

## 5.14 Higher Derivatives อนุพันธ์อันดับสูง

Let $y = f(x)$. We call the derivative of $f'$ the **second derivative** of $f$, denoted by $f''$. Similarly, we can define the third, fourth and higher derivatives of $f$. Also, we denote higher order derivatives of $f$ by

$$f', \quad f'' = (f')', \quad f''' = (f'')', \quad f^{(4)} = (f''')', \quad f^{(5)} = (f^{(4)})', \ldots$$

**Example 5.14.1.** If $f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$, then find the first, second, ... and $n^{th}$-derivatives.

$f'(x) = 12x^3 - 6x^2 + 2x - 4$

$f''(x) = 36x^2 - 12x + 2$

$f'''(x) = 72x - 12$

$f^{(4)}(x) = 72$

$f^{(5)}(x) = 0$ $\qquad f^{(n)} = 0 \; ; n \geq 5$

**Note :** Higher order derivatives of $f$ can also be represented as the followings.

$$f'(x) = \frac{d}{dx}[f(x)]$$

$$f''(x) = \frac{d}{dx}\Big[\frac{d}{dx}[f(x)]\Big] = \frac{d^2}{dx^2}[f(x)]$$

$$f'''(x) = \ldots\ldots\ldots\ldots = \ldots\ldots\ldots$$

$$\vdots \qquad\qquad \vdots$$

In general, we write $\boxed{f^{(n)}(x) = \frac{d^n}{dx^n}[f(x)]}$

If $y = f(x)$, then the higher order derivatives of $f$ can be written as

$$\frac{dy}{dx}, \quad \frac{d^2y}{dx^2}, \quad \frac{d^3y}{dx^3}, \quad \frac{d^4y}{dx^4}, \ldots, \frac{d^ny}{dx^n}, \ldots \text{ or}$$

$$y', \quad y'', \quad y''', \quad y^{(4)}, \ldots, y^{(n)}, \ldots$$

**Example 5.14.2.** Let $f(x) = e^{2x}$. Find $f^{(99)}(x)$.

$f'(x) = 2e^{2x}$

$f''(x) = 2e^{2x} \cdot (2) = 2^2 \cdot e^{2x}$

$f'''(x) = 2^2 e^{2x} (2) = 2^3 \cdot e^{2x}$

$f^{(4)}(x) = 2^3 e^{2x} (2) = 2^4 e^{2x}$

$\vdots$

$f^{(99)}(x) = 2^{99} e^{2x}$

$\vdots$

$$\boxed{f^{(n)}(x) = 2^n e^{2x}}$$

**Example 5.14.3.** Find $\dfrac{d^2y}{dx^2}$ for

1. $y = xe^{-x}$

$\dfrac{dy}{dx} = x\dfrac{d}{dx}(e^{-x}) + e^{-x}\dfrac{dx}{dx}$

$\boxed{= -xe^{-x} + e^{-x}}$

$\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}(-xe^{-x} + e^{-x})$

$= \left[(-x)\dfrac{d}{dx}(e^{-x}) + e^{-x}\dfrac{d(-x)}{dx}\right] + \dfrac{d}{dx}(e^{-x})$

$= xe^{-x} - e^{-x} - e^{-x} = xe^{-x} - 2e^{-x}$ #

2. $y = \sin(x^2 + 1)$

$\dfrac{d^2y}{dx^2}$ ① หา $\dfrac{dy}{dx}$ ② $\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$

① $\dfrac{dy}{dx} = \cos(x^2+1)(2x) = 2x\cos(x^2+1)$

② $\dfrac{d^2y}{dx^2} = 2\left[x\dfrac{d}{dx}(\cos(x^2+1)) + \cos(x^2+1)\dfrac{dx}{dx}\right] = 2\left[(x)(-\sin(x^2+1))\dfrac{d}{dx}(x^2+1) + \cos(x^2+1)\right]$

$= 2\left[(-x)(2x)(\sin(x^2+1)) + \cos(x^2+1)\right]$

$= -4x^2\sin(x^2+1) + 2\cos(x^2+1)$ ##

## 5.15 Approximating Functions

**Definition**

If $f$ can be differentiated $n$ times at $x = a$, then we define the $n^{th}$ **Taylor polynomial** for $f$ about $\boxed{x = a}$ to be

$$\boxed{p_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \ldots + \frac{f^{(n)}(a)(x-a)^n}{n!}}$$

$n! = n(n-1)(\ldots\ldots 3 \cdot 2 \cdot 1$ พหุนามเทเลอร์ กระจายรอบจุด $x=a$

$f(x) = x^2 + 2x + 4$

$$\boxed{f(0.1) = (0.1)^2 + 2(0.1) + 4}$$

$$\boxed{f(x) = \sin x}$$

$f(0.1) = \sin(0.1)$

, cos ___

$e^x$

ln ___

ประมาณฟังก์ชันเหล่านี้
ให้เป็นฟังก์ชันพหุนาม
เพื่อให้คำนวณง่ายๆ

* พหุนาม แค่บวก, ลบ, คูณเป็น
ก็คำนวณค่าได้แล้ว

อนุกรม

$1 + 2 + 3 + \ldots + n$

$1 + 2 + 2^2 + 2^3 + \ldots + 2^n$

เราจะประมาณค่าฟังก์ชันที่ยากๆ ด้วย <u>อนุกรมกำลัง</u> (power series).

Taylor's Series / Maclaurin Seriers

พหุนามเทย์เลอร์ ของ $f(x)$ เมื่อกระจายรอบจุด $x=a$ คือ

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!} + \dots + \frac{f^{(n)}(a)(x-a)^n}{n!}$$

---

$$P_0(x) = f(a)$$

$$P_1(x) = f(a) + f'(a)(x-a)$$

$$P_2(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!}$$

$$P_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \frac{f'''(a)(x-a)^3}{3!}$$

$$\vdots \qquad\qquad \vdots$$

$$P_{100}(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^{(100)}(a)(x-a)^{100}}{100!}$$

แมคลอริน

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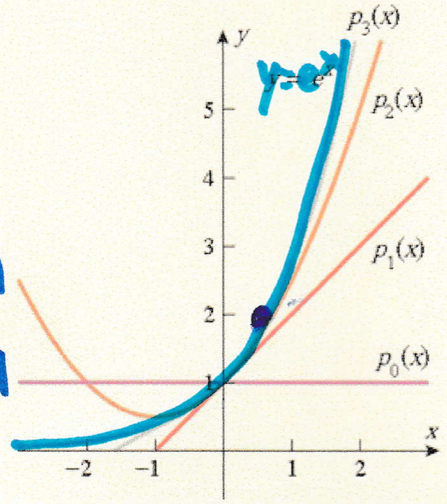
เมื่อ $x=0$

$$P_2(x) = f(0) + f'(0)(x) + \frac{f''(0)(x^2)}{2!}$$

If $f$ can be differentiated $n$ times at $x = 0$, then we define the $n^{th}$ **Maclaurin polynomial** for $f$ to be

$$p_n(x) = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \ldots + \frac{f^{(n)}(0)x^n}{n!}.$$




$\frac{d^n}{dx^n}(e^x) = e^x = f^{(n)}(x)$

$\therefore f^{(n)}(0) = e^0 = 1$

**Example 5.15.1.** Find the Maclaurin polynomials $p_0(x)$ , $p_1(x)$ , $p_2(x)$ and $p_n(x)$ for $e^x$ .

$P_0(x) = f(0) = e^0 = 1$

$P_1(x) = f(0) + f'(0)(x)$

$= 1 + x$

$* \frac{d}{dx}(e^x) = e^x$

$f(x) = e^x$ , $f'(x) = e^x$

$f''(x) = e^x$ $\cancel{f''(0) = 1}$ $f'(0) = 1$

$P_2(x) = f(0) + f'(0)(x) + \frac{f''(0)(x^2)}{2!} = 1 + x + \frac{x^2}{2!}$

$e^x \approx P_n(x) = 1 + x + \frac{x^2}{2!} + \ldots + \frac{x^n}{n!}.$

**Example 5.15.2.** Estimate $e^{0.1}$ using the Maclaurin polynomial $p_2(x)$ .

$e^x \approx P_n(x)$

ต้องการ $P_2(x)$

$e^x \approx P_2(x)$

$= 1 + x + \frac{x^2}{2!}.$

$e^{0.1} \approx 1 + 0.1 + \frac{(0.1)^2}{2!}.$

$= 1 + 0.1 + \frac{0.01}{2}$

$= 1 + 0.1 + 0.005$

$= 1.105$ #

HW

M **Example 5.15.3.** Find the $3^{rd}$ Macluarin polynomials for $f(x) = \sqrt{x+1}$.

$$P_3(x) = f(0) + f'(0)(x) + \frac{f''(0)(x)^2}{2!} + \frac{f'''(0)x^3}{3!}$$

หา
$f'(0)$
$f''(0)$
$f'''(0)$

**Example 5.15.4.** Find the $3^{rd}$ Taylor polynomials for $\sqrt{x}$ about $x = 1$.

T
$$P_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!}$$

หา $f'(1)$
$f''(1)$
$f'''(1)$

**Example 5.15.5.** Estimate $\sqrt{1.1}$ using

1. the $3^{rd}$ Macluarin polynomials of $\sqrt{x+1}$ and
2. the $3^{rd}$ Taylor polynomials of $\sqrt{x}$ about $x = 1$.