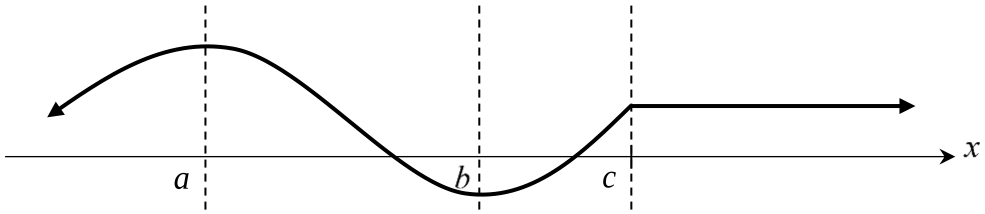
# Chapter 6: Applications of the derivatives

## 6.1 First Derivative and Graphs

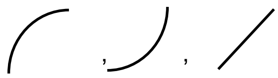
### Increasing and Decreasing Functions

**Example** From the graph below, find Intervals where the given function is increasing, decreasing or constant.




> **Theorem 1** Let $f$ be a function that is continuous on $(a,b)$, and differentiable on $(a,b)$.
> 1. If $f'(x) > 0$ for all $x$ on $(a,b)$, then $f$ is increasing on $(a,b)$.
> 2. If $f'(x) < 0$ for all $x$ on $(a,b)$, then $f$ is decreasing on $(a,b)$.
> 3. If $f'(x) = 0$ for all $x$ on $(a,b)$, then $f$ is constant on $(a,b)$.

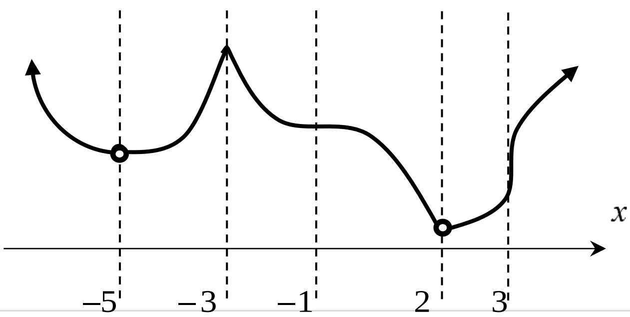
**Increasing and Decreasing function summary:** For the interval $(a,b)$,

| $f'(x)$ | $f(x)$ | Graph of $f$ | Examples |
|---|---|---|---|
| + | Increasing ↗ | Rising ↗ |  |
| – | Decreasing ↘ | Falling ↘ |  |

> **Definition 1** The values of $x$ in the domain of $f$ where $f'(x) = 0$ or where $f'(x)$ does not exist are called the **critical numbers** of $f'(x)$.

**Remark:** The values of $x$ where $f'(x) = 0$ or where $f'(x)$ does not exist are called the **partition numbers** of $f'(x)$. (It does not matter whether $x$ is in the domain of $f$ or not.)

**Example** From the graph below, find



(a) the values of $x$ where $f'(x)=0$

(b) the values of $x$ where $f'(x)$ does not exist

(c) the partition numbers of $f'(x)$

(d) the domain of $f$

(e) the values of $x$ in the domain of $f$ where $f'(x)=0$

(f) the values of $x$ in the domain of $f$ where $f'(x)$ does not exist

(g) the critical numbers of $f'(x)$

**Example** Find the critical numbers of $f'(x)$, the intervals on which $f$ is increasing and those on which $f$ is decreasing, for

(a) $f(x)=x^2-4x+5$

(b) $f(x)=x^3-3x$

(c) $f(x)=\sqrt{4-x^2}$ .

## Local Extrema

**Definition 2** Let $c$ be a value in the domain of the function $f$ that is continuous on $(a,b)$.

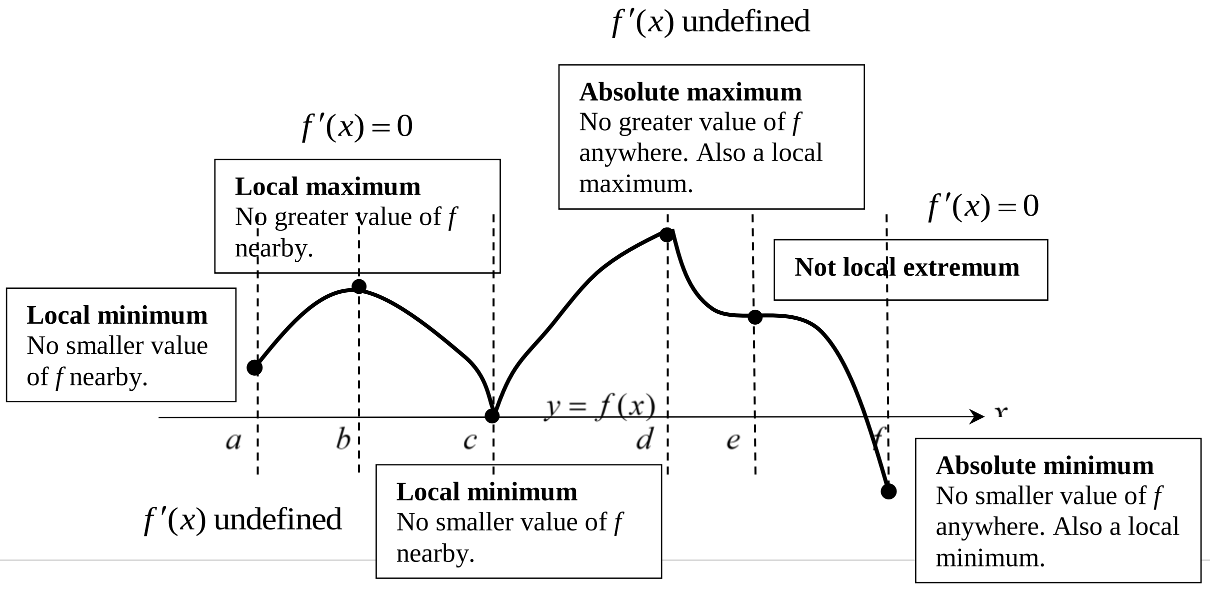
(1) We call $f(c)$ a **local maximum** if there exists an interval $(m,n)$ containing $c$ where $f(x)\le f(c)$ for all $x$ in $(m,n)$.

(2) We call $f(c)$ a **local minimum** if there exists an interval $(m,n)$ containing $c$ where $f(x)\ge f(c)$ for all $x$ in $(m,n)$.

(3) We call $f(c)$ a **local extremum,** if $f(c)$ is a **local maximum** or a **local minimum**.

**Theorem 2** Let $f$ be a function that is continuous on $(a,b)$, and $c$ is a value in $(a,b)$ where $f(c)$ is a local extremum, then $c$ is a critical number.

**Remark:** Not all critical numbers of $f'(x)$ are local extrema.

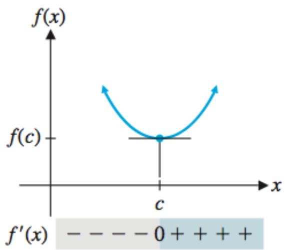


## First-Derivative Test

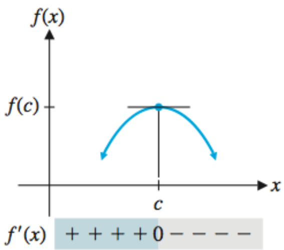
Let $c$ be a critical number of the function $f'(x)$ that is continuous on $(a,b)$ containing $c$. If $f$ is differentiable on the interval, except possibly at $c$, then $f(c)$ can be classified as:

1. a **local minimum**, if $f'(x)$ changes from negative (or zero) to positive (or zero) at $c$. ↘ ↗
2. a **local maximum**, if $f'(x)$ changes from positive (or zero) to negative (or zero) at $c$. ↗ ↘
3. neither a max nor a min if $f'(x)$ is positive on both sides of $c$ ↗ ↗ or negative on both sides of $c$. ↘ ↘

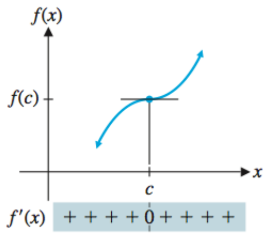
$f'(c)=0$: **Horizontal Tangent**



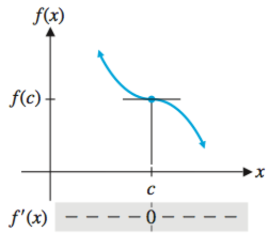
(A) f(c) is a local minimum



(B) f(c) is a local maximum

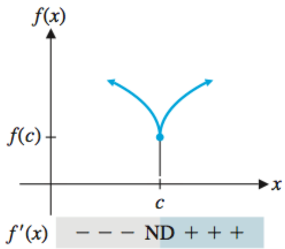


(C) f(c) is neither a local maximum nor a local minimum

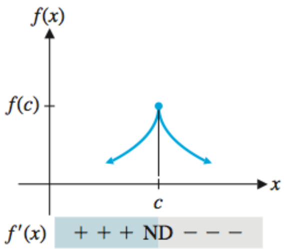


(D) f(c) is neither a local maximum nor a local minimum

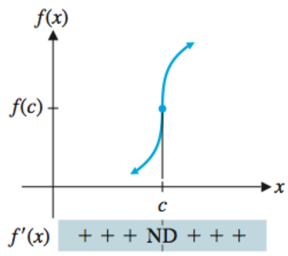
$f'(c)$ **is not defined, but** $f(c)$ **is defined.**



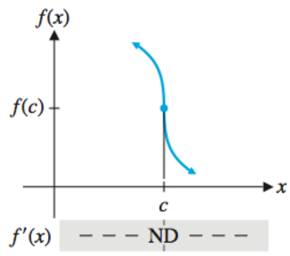
(E) f(c) is a local minimum



(F) f(c) is a local maximum



(G) f(c) is neither a local maximum nor a local minimum



(H) f(c) is neither a local maximum nor a local minimum

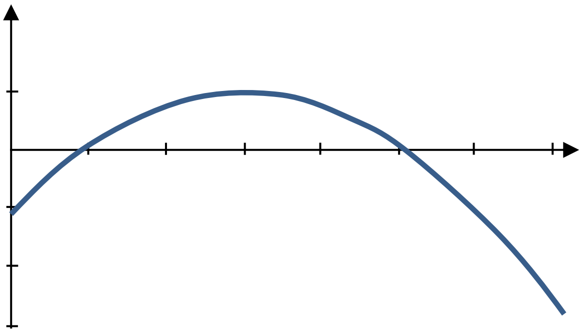
**Example** Find the local extrema (if exists) for the following functions, and determine the type of extrema.

(a) $f(x)=x^2-4x+5$

(b) $f(x)=x^3-3x$

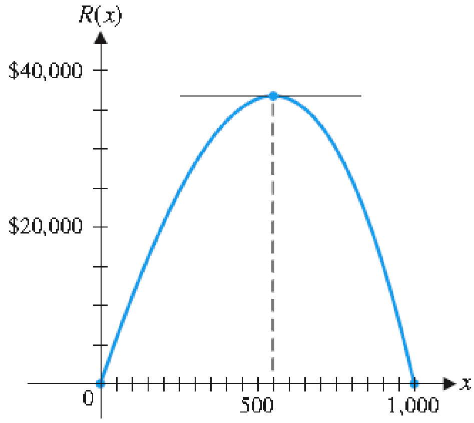
## Applications

**Example** The graph in the figure approximates the rate of change of the price of eggs over a 70 month period, where $E(t)$ is the price of a dozen of eggs (in dollars), and $t$ is the time in months. Determine when the price of eggs was rising or falling.



$0 < x < 70$ and $-0.03 < y < 0.015$

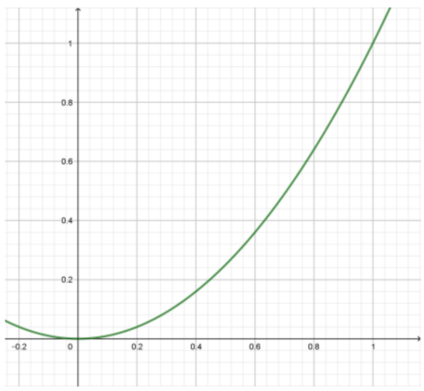
**Example (Revenue Analysis)** The graph of the total revenue $R(x)$ (in dollars) from the sale of $x$ bookcases is shown below. When would the marginal revenue be positive, and when would it be negative?



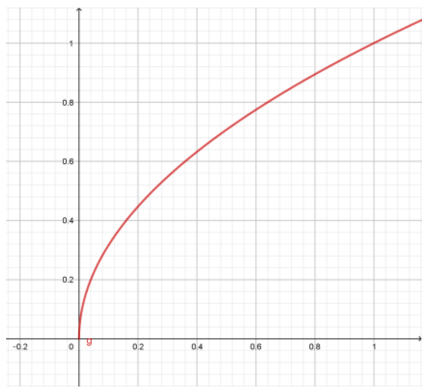

# 6.2 Second Derivative and Graphs

## Using Concavity as a graphing Tool

**Example** Discuss the relationship between the values of the derivatives of $f$ and $g$ and the shapes of their graphs.



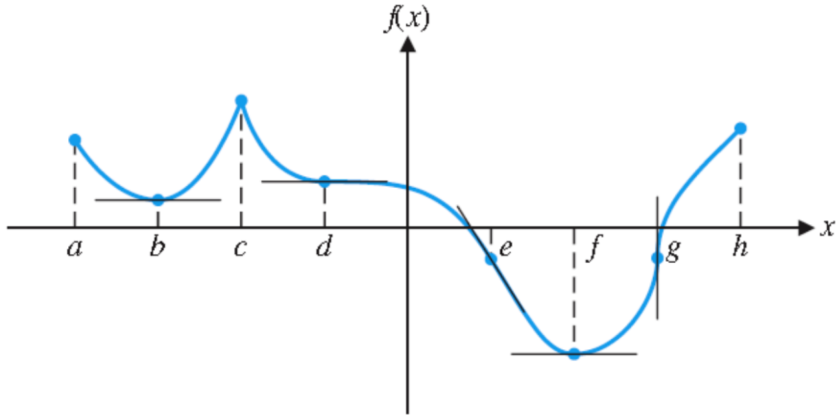
(a) $f(x) = x^2$



(b) $f(x) = \sqrt{x}$

**Definition 3** The graph of function $f$ is **concave upward** on the interval $(a,b)$ if $f'(x)$ is increasing on $(a,b)$ and is **concave downward** on the interval $(a,b)$ if $f'(x)$ is decreasing on $(a,b)$.

**Example** From the graph of $f$ below, find




(a) intervals where the graph of $f$ is concave upward

(b) intervals on which $f'(x)$ is increasing

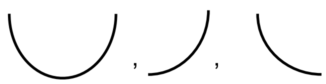
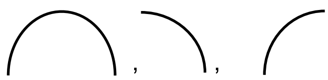
(c) intervals on which $f''(x) > 0$

(d) intervals where the graph of $f$ is concave downward

(e) intervals on which $f'(x)$ is decreasing

(f) intervals on which $f''(x) < 0$

**Concavity summary:** For the interval $(a,b)$,

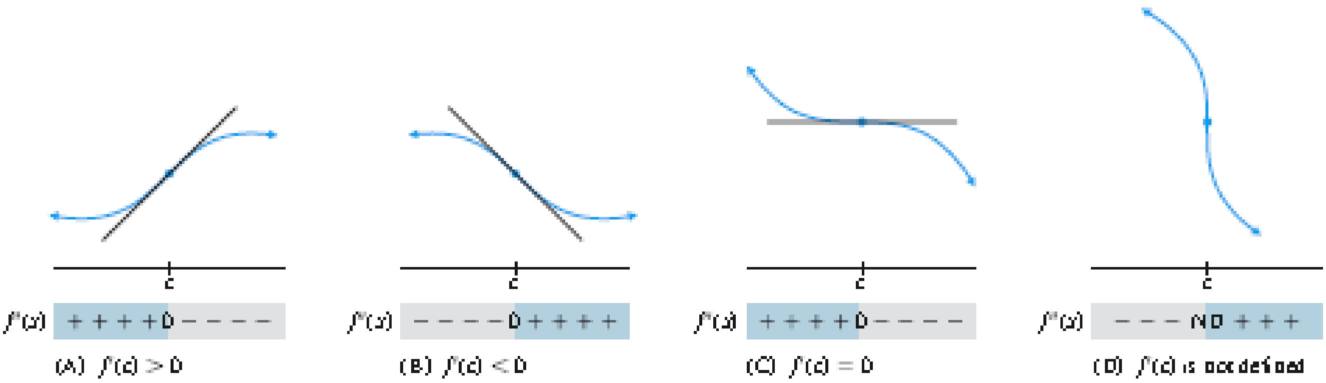
| $f''(x)$ | $f'(x)$ | Graph of $f$ | Examples |
|---|---|---|---|
| + | Increasing | Concave upward |  |
| – | Decreasing | Concave downward |  |

**Example** Find the intervals where the graph of $f(x) = x^3 + 24x^2 + 15x - 12$ is concave up or concave down.

**Example** Find the intervals where the graph of $f(x)=\ln\left(x^2-4x+5\right)$ is concave up or concave down.

## Finding Inflection Points

**Definition 4** If the graph of $y=f(x)$ changes concavity at $x=c$, then the point $(c, f(c))$ is called an **inflection point**.




**Remark:** The values of $x$ where $f''(x)=0$ or where $f''(x)$ does not exist are called the **partition numbers** of $f''(x)$. (It does not matter whether it is in the domain of $f$ or not.)

**Theorem 3** If $y=f(x)$ is continuous on $(a,b)$ and has an inflection point at $(c, f(c))$, then either $f''(c)=0$ or $f''(c)$ does not exist.

**Example** Find the inflection points of

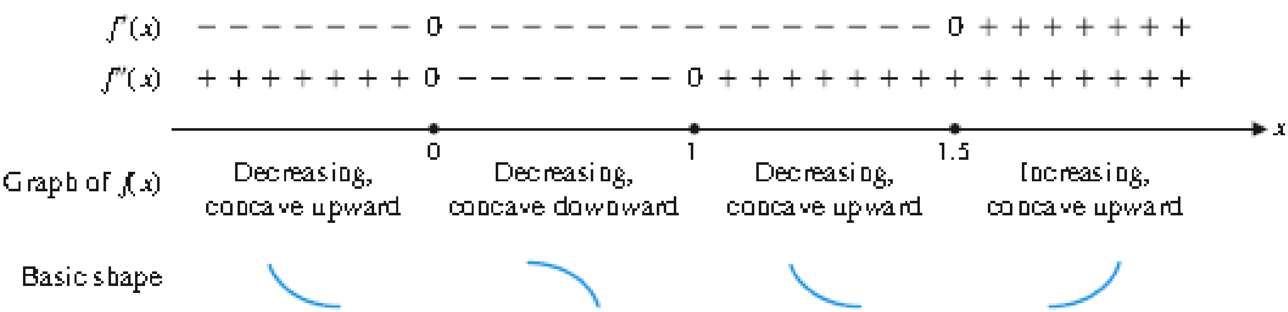
(a) $f(x)=x^4$

(b) $f(x) = x^3 - 3x$

## 6.3. Curve-Sketching Techniques

In this course, we concentrate on the functions that are continuous at every point in the set of real number and have no corner points. (Thus, partition numbers for $f'(x)$ and critical numbers are the same.)

| |
|---|
| **Procedure** *Graphing Strategy* |
| **Step 1** ( $f'(x)$ *analysis*) Find the partition numbers for $f'(x)$. |
| Construct a sign chart for $f'(x)$ to determine the intervals on which $f$ is increasing and decreasing. |
| **Step 2** ( $f''(x)$ *analysis*) Find the partition numbers for $f''(x)$. |
| Construct a sign chart for $f''(x)$ to determine the intervals on which $f$ is concave upward and concave downward. |
| **Step 3** (*Sketch the graph of* $f$ ) Locate the points whose $x$ value is a partition number (for $f'(x)$ or for $f''(x)$.) |
| Sketch in what you know from the previous steps. |



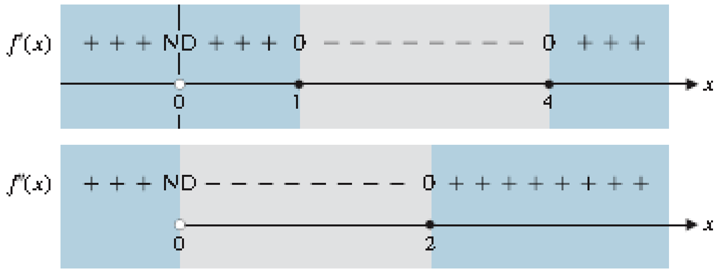

**Example** Sketch the graph of $y = x^3 - 3x + 2$.

**Example** Sketch the graph of $y = x^4 + 4x^3 + 14$.

**Example** Use the given information to sketch the graph of $f$ .

(a)

| $x$ | $-3$ | $0$ | $1$ | $2$ | $4$ | $5$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $-4$ | $0$ | $2$ | $1$ | $-1$ | $0$ |




(b)

$f(0) = -2, f(1) = 0, f(2) = 4;$

$f'(0) = 0, f'(2) = 0, f'(1)$ is not defined;

$f'(x) > 0$ on $(0, 1)$ and $(1, 2)$;

$f'(x) < 0$ on $(-\infty, 0)$ and $(2, \infty)$;

$f''(1)$ is not defined;

$f''(x) > 0$ on $(-\infty, 1)$;

$f''(x) < 0$ on $(1, \infty)$

## 6.4 Optimization

**Procedure** *Strategy for Solving Optimization Problems*

**Step 1** Introduce variables, look for relationships among these variables, and construct a math model of the form Maximize (or minimize) $f(x)$ on a closed interval $[a,b]$.

**Step 2** Find the critical numbers of $f'(x)$.

**Step 3** Compute the value of $f(x)$ when $x$ is a critical number or an endpoint of the interval, says $a$ and $b$.

**Step 4** Find the maximum (minimum) solution of $f(x)$ on the interval $[a,b]$ from all critical numbers and endpoints.

**Step 5** Use the solution to the mathematical model to answer all the questions asked in the problem.

**Remark:** The maximum (minimum) value of $f(x)$ is called the **maximum (minimum) value**.

The value of $x$ giving the maximum (minimum) value of $f(x)$ is called a **maximum (minimum) solution**.

We say that these $x$ values that are maximum (minimum) solutions **maximize (minimize)** $f(x)$, or we can say that $f(x)$ **is maximized (minimized)** at these $x$ values.

**Example** A company manufactures and sells $x$ television sets per month. The price-demand equation is $p(x) = 200 - \frac{x}{50}$ for 0 ≤ $x$ ≤ 6,000. Find the production level that will maximize the revenue, the maximum revenue, and the price that the company needs to charge at that level.

**Example** The cost function of opening $x$ out of 60 rooms in a dormitory for rent is $C(x)=x(100-x)$ dollars per month. Find the number of rental rooms that minimizes the cost if there must be at least 10 rooms for rent.

**Exercise**

1. Let $f(x)=x^3-3x^2+1$.

   (a) Find the critical numbers of $f'(x)$ and the intervals on which $f$ is increasing or decreasing.

   (b) Find the local extrema (if exists) of $f$, and determine the type of extrema.

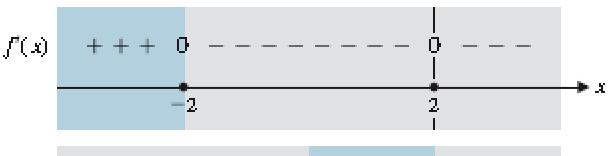
   (c) Find the intervals where the graph of $f$ is concave up or concave down.

   (d) Find the inflection points of $f$.

   2. Use the given information to sketch the graph of $f$.

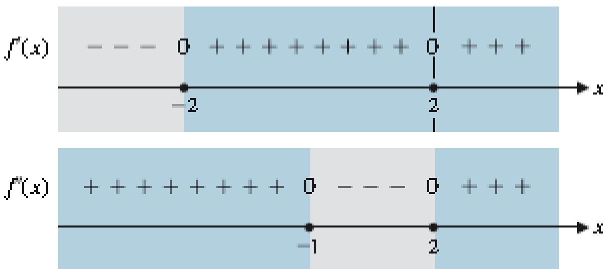
   (a)

| $x$ | $-4$ | $-2$ | $-1$ | $0$ | $2$ | $4$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $0$ | $3$ | $1.5$ | $0$ | $-1$ | $-3$ |




| $x$ | $-4$ | $-2$ | $-1$ | $0$ | $2$ | $4$ |
|---|---|---|---|---|---|---|
| $f(x)$ | $0$ | $-2$ | $-1$ | $0$ | $1$ | $3$ |

   (b)



(c)

$f(0) = 2, f(1) = 0, f(2) = -2;$

$f'(0) = 0, f'(2) = 0;$

$f'(x) > 0$ on $(-\infty, 0)$ and $(2, \infty);$

$f'(x) < 0$ on $(0, 2);$

$f''(1) = 0;$

$f''(x) > 0$ on $(1, \infty);$

$f''(x) < 0$ on $(-\infty, 1)$

3. A company manufactures and sells $x$ television sets per month. The monthly cost and price-demand equations are $C(x) = 60000 + 60x$ and $p(x) = 200 - \frac{x}{50}$ for $0 \le x \le 6{,}000$. Find the production level that will maximize the profit, the maximum profit, and the price that the company needs to charge at that level.