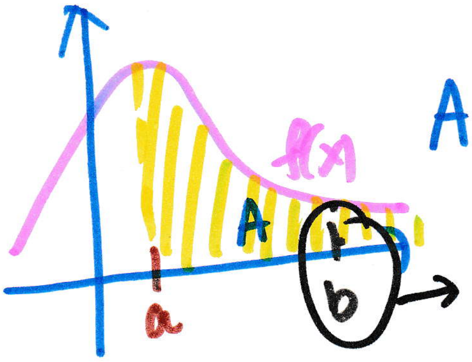
Improper integral $\boxed{\text{3 Types}}$

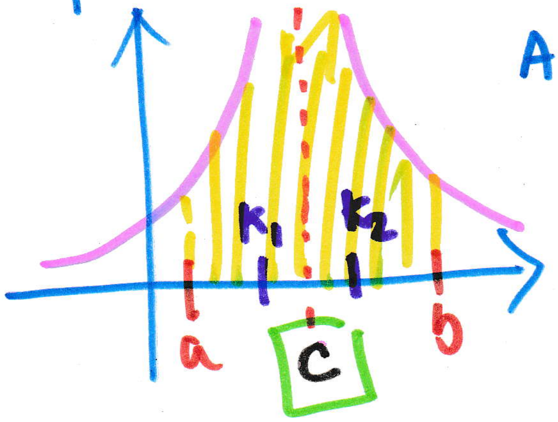
Type 1 ขอบเขตเป็นอนันต์



$$A = \int_a^{+\infty} f(x)\,dx$$

$$= \lim_{b \to +\infty} \int_a^b f(x)\,dx$$

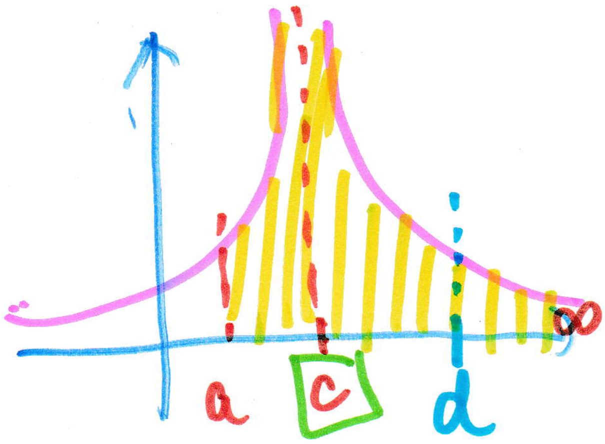
Type 2 มีจุดในฟังก์ชันที่ทำให้ฟังก์ชันไม่ต่อเนื่อง แบบมี V.A.



$$A = \int_a^{\boxed{c}} f(x)\,dx + \int_{\boxed{c}}^b f(x)\,dx$$

$$= \lim_{k_1 \to c^-} \int_a^{k_1} f(x)\,dx + \lim_{k_2 \to c^+} \int_{k_2}^b f(x)\,dx$$

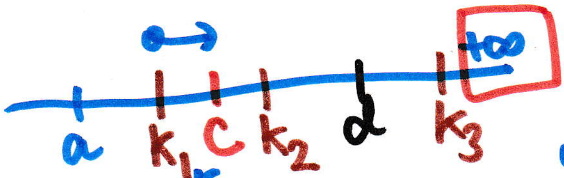
Type 3 1 + 2



$$A = \int_a^{+\infty} f(x)\,dx$$

$$= \underbrace{\boxed{\int_a^c f(x)\,dx}}_{\text{Type 2}} + \underbrace{\boxed{\int_c^{+\infty} f(x)\,dx}}_{\text{Type 3}}$$

$$= \underbrace{\int_a^c f(x)\,dx}_{\text{Type 2}} + \underbrace{\int_c^d f(x)\,dx}_{\text{Type 2}} + \underbrace{\int_d^{+\infty} f(x)\,dx}_{\text{Type 1}}$$



$$\therefore A = \lim_{k_1 \to c^-} \int_a^{k_1} f(x)\,dx + \lim_{k_2 \to c^+} \int_{k_2}^d f(x)\,dx + \lim_{k_3 \to +\infty} \int_d^{k_3} f(x)\,dx$$

Example 7.26 Compute $\int_0^1 \frac{1}{1-x}\,dx$.

Type 2 : f discontinuous at $x=1$

$$\int_0^1 \frac{1}{1-x}\,dx = \lim_{K\to 1^-} \int_0^K \frac{1}{1-x}\,dx$$

$u = 1-x$; $du = -dx$

$\int \frac{1}{1-x}\,dx = -\int \frac{1}{u}\,du = -\ln|u| + C$

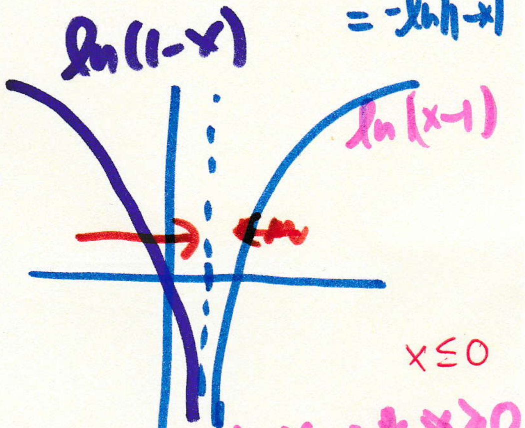
$= -\ln|1-x|$

$$= \lim_{K\to 1^-} \left[-\ln|1-x|\right]_0^K$$

$$= \lim_{K\to 1^-} \left[-\ln(1-K) + \ln|1|\right]$$

$$= +\infty$$

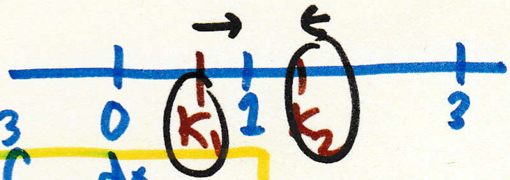
$\therefore \int_0^1 \frac{1}{1-x}\,dx$ diverges.



$$|1-x| = \begin{cases} 1-x & ; \cancel{1-x>0} \quad x\le 0 \\ x-1 & ; \cancel{1-x<0} \quad x-1>0 \end{cases}$$

Example 7.27 Compute $\int_0^3 \frac{dx}{(x-1)^{2/3}}$.

Type 2 : f discontinuous at $x=1$



$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}$$

$$= \boxed{\lim_{K_1\to 1^-} \int_0^{K_1} \frac{dx}{(x-1)^{2/3}}} + \boxed{\lim_{K_2\to 1^+} \int_{K_2}^3 \frac{dx}{(x-1)^{2/3}}} \quad (*)$$

$$\lim_{K_1\to 1^-} \int_0^{K_1} \frac{dx}{(x-1)^{2/3}} = \lim_{K_1\to 1^-} \left[3(x-1)^{\frac{1}{3}}\right]_0^{K_1}$$

$$= \lim_{K_1\to 1^-} \left[3(K_1-1)^{\frac{1}{3}} - 3(0-1)^{\frac{1}{3}}\right]$$

$$= \lim_{K_1\to 1^-} \left[3(K_1-1)^{\frac{1}{3}} + 3\right] = 3$$

$$\lim_{K_2\to 1^+} \int_{K_2}^3 \frac{dx}{(x-1)^{2/3}} = \lim_{K_2\to 1^+} \left[3(x-1)^{\frac{1}{3}}\right]_{K_2}^3$$

$$= \lim_{K_2\to 1^+} \left[3\sqrt[3]{2} - 3(K_2-1)^{\frac{1}{3}}\right]$$

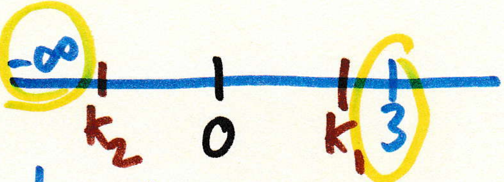
$$= 3\sqrt[3]{2}$$

$(*) = 3 + 3\sqrt[3]{2}$

$\therefore$ converges.

Improper Integral: Third type

Additional Example $\int_{-\infty}^{3} \frac{1}{(x-3)(x-4)}\,dx$



$$\int_{-\infty}^{3} \frac{1}{(x-3)(x-4)}\,dx = \int_{-\infty}^{0} \frac{1}{(x-3)(x-4)}\,dx + \int_{0}^{3} \frac{1}{(x-3)(x-4)}\,dx$$

$$= \boxed{\lim_{k_2 \to -\infty} \int_{k_2}^{0} \frac{1}{(x-3)(x-4)}\,dx} + \boxed{\lim_{k_1 \to 3^-} \int_{0}^{k_1} \frac{1}{(x-3)(x-4)}\,dx} \quad \text{---} (*)$$

ทำ $\int \frac{1}{(x-3)(x-4)}\,dx$

$$\frac{1}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$$

$$1 = A(x-4) + B(x-3)$$

(แทน $x=4$ $\boxed{1 = B}$

$x=3$ $1 = -A \Rightarrow \boxed{A=-1}$)

$$1 = Ax - 4A + Bx - 3B$$

$$1 = (A+B)x + (-4A-3B)$$

เทียบ สปส. $A+B = 0$, $-4A-3B = 1$ $\Big\}$ $A=-1$, $B=1$

$$\int \frac{1}{(x-3)(x-4)}\,dx = \int -\frac{1}{x-3}\,dx + \int \frac{1}{x-4}\,dx$$

$$= -\ln|x-3| + \ln|x-4| + C$$

$(*) =$

$$\boxed{\lim_{k_2 \to -\infty} \int_{k_2}^{0} \frac{1}{(x-3)(x-4)}\,dx} = \lim_{k_2 \to -\infty} \Big[-\ln|x-3| + \ln|x-4|\Big]_{k_2}^{0}$$

$$= \lim_{k_2 \to -\infty} \Big[-\ln|3| + \ln 4 + \underline{\ln|k_2-3| - \ln|k_2-4|}\Big]$$

$$= \lim_{k_2\to-\infty} \left[-\ln 3 + \ln 4 + \ln\left|\frac{k_2-3}{k_2-4}\right|\right] \quad (**)$$

$\to 1$

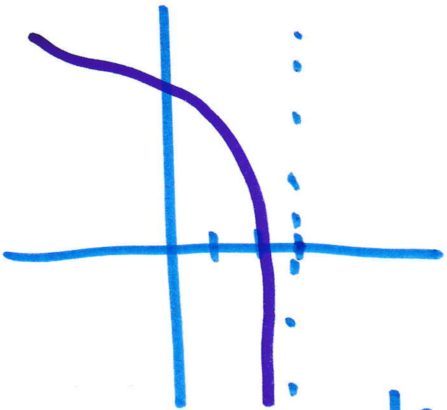
$$\lim_{k_2\to-\infty} \ln\left|\frac{k_2-3}{k_2-4}\right| = \ln\left|\underbrace{\lim_{k_2\to-\infty} \frac{k_2-3}{k_2-4}}\right| = 0$$

$$(**) = -\ln 3 + \ln 4$$

$$\boxed{\lim_{k_1\to 3^-} \int_0^{k_1} \frac{1}{(x-3)(x-4)}\,dx} = \lim_{k_1\to 3^-} \left[-\ln|x-3| + \ln|x-4|\right]_0^{k_1}$$

$$= \lim_{k_1\to 3^-} \left[-\ln|k_1-3| + \underbrace{\ln|k_1-4|}_{\ln 1} + \ln 3 - \ln 4\right]$$

$$= +\infty$$



$$\therefore \lim_{k_1\to 3^-} \int_0^{k_1} \frac{1}{(x-3)(x-4)}\,dx \text{ diverges}$$

$$|k_1-3| = \begin{cases} k_1-3 \,; & k_1-3 \geq 0 \\ -(k_1-3) \,; & k_1-3 < 0 \end{cases}$$

$$\therefore \lim_{k_1\to 3^-} -\ln|k_1-3| = \lim_{k_1\to 3^-} -\ln(3-k_1)$$

$$\therefore \int_{-\infty}^{3} \frac{1}{(x-3)(x-4)}\,dx \text{ diverges.}$$

$$x^2 + 2x + 1 = 0 \quad — \text{★}$$

$$\boxed{x = -1}$$

$x = -1$ เป็นคำตอบของสมการ ★ เพราะ

$$(-1)^2 + 2(-1) + 1 = 0$$

$$\boxed{F'(x) = f(x)}$$

$$\int F'(x)\,dx = \int f(x)\,dx$$

$$\int f(x)\,dx = F(x)\cancel{dx} + C$$

ให้ y เป็นฟังก์ชันของ x

<u>จงหา</u> y ที่สอดคล้องกับ

$$y' = f(x)$$

<u>Ex</u> จงหา y ที่สอดคล้องกับ

$$y' = \cos x$$

y คือ $\sin x$, $\sin x + 5$, ...

$$\boxed{y = \sin x + C}$$

Ex หา y ที่สอดคล้องกับ $\boxed{y' + (\cos x)y = e^x}$

Ex $y'' + 2y' + (2x)y = 0$

สมการเชิงอนุพันธ์ อันดับ 2

order (อันดับ) ของสมการเชิงอนุพันธ์ คือ อนุพันธ์อันดับสูงสุดที่ปรากฏในสมการ

- สมการเชิงอนุพันธ์สามัญ (Ordinary Differential Eq^{n}) ODEs
- สมการเชิงอนุพันธ์ย่อย (Partial Differential Eq^{n}) PDEs

8 Differential Equations

Now consider an equation

$$y' = f(x). \tag{8.2}$$

One could ask: what is a function y (of x) that fits the equation? Or, equivalently, what is an antiderivative of $f(x)$? Hence, if $F(x)$ is an antiderivative of $f(x)$, then $y = F(x)$ is a solution of (8.2) and $y = F(x) + c$ is a general form of all solutions. The equation (8.2) is an example of a *differential equation* , and its ***general solution*** is given by $y = F(x) + c$.

In general, differential equations can be a lot more complicated than (8.2). They may contain higher order derivatives or even products of derivatives.

> DEFINITION A **differential equation** is an equation that involves one or more derivatives of unknown functions. The **order** of a differential equation is the order of the highest derivatives in the equation.

A differential equation with one unknown is called an *ordinary differential equation* (ODE). A differential equation with two or more unknowns is called a *partial differential equation* (PDE). Here, we **only** deal with ordinary differential equations, so we shall omit the term "ordinary".

Example 8.1 Here are some examples of differential equations.

1. $ye^x = 1$

   is of order zero because y is the highest derivative of y itself, which is of order zero.

2. $\dfrac{dy}{dx} = \cos x$

   is of order ......1...... because ...the highest derivative is of degree 1.

3. $y - y' = (\ln x + \tan 2x)y'' + \sqrt{x + y}$

   is of order ......2...... because 2

ฟังก์ชันที่สอดคล้องกับสมการ

DEFINITION A **solution** of a differential equation is a function that satisfies the equation. A **general solution** of a differential equation is an expression that contains all possible solutions. คำตอบทั่วไป

Example 8.2

1. The constant function $y = 1$ is a solution of the equation $\boxed{\frac{dy}{dx} = 0}$ because $\frac{d}{dx}(1) = 0$, but it is not the only solution.

   → หา y

   We also know that the derivative of any constant function is zero.

   Hence, the general solution of $\frac{dy}{dx} = 0$ is $\boxed{y = c.}$

   ถ้า $y = e^x$, $\frac{dy}{dx} = e^x = y$

2. $y = e^x$ is a solution of the equation $\frac{dy}{dx} = y$ because ....

   Note that $y = e^x + c$ is not a solution of the equation unless $c = 0$.

   The general solution of this equation is ...... $y = ce^x$ ......

   because ...... $\frac{dy}{dx} = ce^x = y$ ......

3. $y = x + \frac{1}{x}$ is a solution of the equation $x\frac{dy}{dx} + y = 2x$. $\quad y = x + \frac{1}{x} \Rightarrow \frac{dy}{dx} = 1 - \frac{1}{x^2}$

   because ... $x\left(1 - \frac{1}{x^2}\right) + x + \frac{1}{x} = x - \cancel{\frac{1}{x}} + x + \cancel{\frac{1}{x}} = 2x$ ......

   It is not as easy as earlier examples to guess what the general solution should be. However, it might look easier if we rewrite the equation as

$$\frac{d}{dx}(xy) = 2x.$$

   The general solution of this equation is ....

   because ....

(8.2)(2) $y = e^x$ เป็นผลเฉลยของ $\frac{dy}{dx} = y$

$y = e^x + C$ เป็นผลเฉลยของ $\frac{dy}{dx} = y$ ไหม?

$$\frac{dy}{dx} = e^x \neq e^x + c = y$$

แต่ general solution $\quad y = ce^x$

$$\frac{dy}{dx} = ce^x = y$$

คำตอบเฉพาะ (particular solution)

เงื่อนไขเริ่มต้นที่แตกต่าง เราจะหาค่า C ได้ที่สอดคล้องกับเงื่อนไข

initial condition $\quad \boxed{y(0) = 1} \Rightarrow$ ค่า C นั้นต้องเป็นอะไร?

$\downarrow$

C ที่สอดคล้อง

$$1 = ce^{(0)}$$

$\Rightarrow \boxed{c = 1} \quad \Rightarrow \boxed{y = e^x}$ คำตอบเฉพาะ เงื่อนไขเริ่มต้น $y(0) = 1$

Problem 8.1 Verify that both $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of the differential equation $y'' = y$. How about their linear combination $y = c_1e^x + c_2e^{-x}$?

Solution. จะแสดง $y_1 = e^x$ เป็นคำตอบของ $y'' = y$

$y_1' = e^x$

$y_1'' = e^x$

$\therefore y_1'' = e^x = y$ เป็น

$y = e^{-x}$;

$y_2' = -e^{-x}$

$y_2'' = e^{-x} = y$

เป็น

$y = c_1e^x + c_2e^{-x}$

$y' = c_1e^x - c_2e^{-x}$

$y'' = c_1e^x + c_2e^{-x} = y$ เป็น

> DEFINITION An **initial-value problem** is a differential equation with initial conditions (conditions that solutions of the differential equation needs to meet, conditions on values of solutions). A **particular solution** of a differential equation is a solution of the equation that satisfies the initial conditions. That is, a particular solution will not contain arbitrary constants.

Example 8.3 The second-order differential equation

$$y'' = -y$$

has a general solution

$$y = c_1 \cos x + c_2 \sin x$$

where c_1 and c_2 are constants (Verify!). We may add two conditions on the values of y. For instance, set

$$y(0) = 1 \quad \text{and} \quad y(\frac{\pi}{2}) = 2. \tag{8.3}$$