สมการเชิงอนุพันธ์ Differential Equation.

$$y' - 4y = \cos x$$

$$x\frac{d^2y}{dx^2} + 4\cos x\, y = \sin x$$

$y = xe^x$ $\boxed{y'' - y = e^x}$

y เป็นคำตอบของสมการเชิงอนุพันธ์

$$\left.\begin{aligned} y'' &= \underline{\hspace{3cm}} \\ y &= \underline{\hspace{3cm}} \end{aligned}\right\}$$

คำตอบของสมการเชิงอนุพันธ์

① คำตอบทั่วไป (General solution)

② คำตอบเฉพาะ (particular solution)

DEFINITION An **initial-value problem** is a differential equation with initial conditions (conditions that solutions of the differential equation needs to meet, conditions on values of solutions). A **particular solution** of a differential equation is a solution of the equation that satisfies the initial conditions. That is, a particular solution will not contain arbitrary constants.

Example 8.3 The second-order differential equation

$$y'' = -y$$

has a general solution

$$\boxed{y = c_1 \cos x + c_2 \sin x}$$

where c_1 and c_2 are constants (Verify!). We may add two conditions on the values of y. For instance, set

$$y(0) = 1 \quad \text{and} \quad y(\frac{\pi}{2}) = 2. \tag{8.3}$$

CHECK ตรวจการ $y'' = -y$? $y = C_1 \cos x + C_2 \sin x$

หา y'' : $y' = -C_1 \sin x + C_2 \cos x$

$$y'' = -C_1 \cos x - C_2 \sin x$$

$$= -\underline{[C_1 \cos x + C_2 \sin x]}$$

$$= -y$$

$\therefore$ $y = C_1 \cos x + C_2 \sin x$ เป็นคำตอบของ $y'' = -y$

$y(0) = 1$: $1 = C_1(1) + C_2(0) \Rightarrow 1 = C_1$

$y(\frac{\pi}{2}) = 2$: $2 = C_1 \cancel{\cos(\frac{\pi}{2})} + C_2 \sin(\frac{\pi}{2}) \Rightarrow 2 = C_2$

$\therefore$ คำตอบเฉพาะเมื่อ $y(0)=1, y(\frac{\pi}{2})=2$ คือ

$$\boxed{y = \cos x + 2 \sin x}$$

#

Remarks.

1. The definition of solutions is a bit loose. Any solution must be well-defined on some certain interval, where solutions are sought.

   Consider the equation

   $$x^2\frac{dy}{dx} = x.$$

   - All solutions of this equation are given by $\underline{y = \ln|x| + c}$ on $(-\infty, 0) \cup (0, \infty)$. But this equation has no solutions on $[-1, 1]$.
   - If we want solutions on $(0, \infty)$, the general solution is given by $y = \ln x + c$. On the other hand, the general solution is given by $y = \ln(-x) + c$ on $(-\infty, 0)$.

   $$|x| = \begin{cases} x ; x \geq 0 \\ -x ; x < 0 \end{cases}$$

   Consider the equation

   $$yy' + x = 0.$$

   - $y = \sqrt{-1 - x^2}$ cannot be a solution even though it satisfies the equation (verify!).

2. In general, a differential equation might not have solutions even it is of first order, e.g., the equation

   $$(y')^2 + y^2 + 1 = 0$$

   does not have solutions since (why?) ....

Problem 8.2 Find the particular solution of the differential equation

$$2y\frac{dy}{dx} = 3x^2, \quad \underbrace{y(0) = -3}_{\text{initial value problem}}$$

provided that its general solution is given implicitly by

$$\boxed{y^2 = x^3 + c.} \quad \text{— } (*)$$

Note that this particular solution is a solution on $(-\infty, 0]$.

Solution. initial condition: $y(0) = -3$

แก้ (*) เพื่อหาค่า c

$$(-3)^2 = (0)^2 + C$$

$$\boxed{C = 9}$$

$\therefore$ particular solution คือ

$$y^2 = x^3 + 9 \qquad \#$$

ภายใต้เงื่อนไข $y(0) = -3$.

① $\frac{dy}{dx} = \frac{x}{y} \Longrightarrow \boxed{\frac{\heartsuit}{y}}\frac{dy}{dx} = \boxed{\frac{\text{✿}}{x}}$

$$\underbrace{y}_{h(y)}\frac{dy}{dx} = \underbrace{x}_{g(x)}$$

$h(y) = y$

$g(x) = x$

② $\boxed{x}\frac{dy}{dx} = \boxed{-y^3} \Rightarrow \underbrace{\left(-\frac{1}{y^3}\right)}_{h(y)}\frac{dy}{dx} = \underbrace{\frac{1}{x}}_{g(x)}$

③ $\boxed{}\frac{dy}{dx} = \boxed{y - \frac{y}{x}} \Rightarrow \frac{dy}{dx} = y\left(1 - \frac{1}{x}\right)$

$$\underbrace{\frac{1}{y}}_{h(y)}\frac{dy}{dx} = \underbrace{\left(1 - \frac{1}{x}\right)}_{g(x)}$$

④ $\frac{y}{x}\frac{dy}{dx} = x\sqrt{1+y^2} \Rightarrow \underbrace{\frac{y}{\sqrt{1+y^2}}}_{h(y)}\frac{dy}{dx} = \underbrace{x^2}_{g(x)}$

⑤ $\frac{x^2}{1+y^2} - \frac{dy}{dx} = 0 \Rightarrow \frac{x^2}{1+y^2} = \frac{dy}{dx}$

$$(1+y^2)\frac{dy}{dx} = x^2$$

8.1 Separable Equations สมการแบบแยกตัวแปร

First-order separable equations are differential equations that can be expressed in the form

$$h(y)\frac{dy}{dx} = g(x). \tag{8.6}$$

โยน y ไว้ฝั่งหนึ่ง
โยน x ไว้อีกฝั่งหนึ่ง

$h(y)dy = g(x)dx$: รูป differential

Example 8.4

Original equation	Separable form	$h(y)$	$g(x)$
① $\frac{dy}{dx} = \frac{x}{y}$	$y\frac{dy}{dx} = x$	y	x
② $x\frac{dy}{dx} = -y^3$	$\left(-\frac{1}{y^3}\right)\frac{dy}{dx} = \frac{1}{x}$	$-\frac{1}{y^3}$	$\frac{1}{x}$
③ $\frac{dy}{dx} = y - \frac{y}{x}$	$\frac{1}{y}\frac{dy}{dx} = 1 - \frac{1}{x}$	$\frac{1}{y}$	$1-\frac{1}{x}$
④ $\frac{y}{x}\frac{dy}{dx} = x\sqrt{1+y^2}$	$\frac{y}{\sqrt{1+y^2}}\frac{dy}{dx} = x^2$	$\frac{y}{\sqrt{1+y^2}}$	x^2
⑤ $\frac{x^2}{1+y^2} - \frac{dy}{dx} = 0$	$(1+y^2)\frac{dy}{dx}$	$1+y^2$	x^2

$$h(y)dy = g(x)dx$$

อินทิเกรตทั้งสองข้าง

$$\int h(y)dy = \int g(x)dx$$

We summarise the method for solving a separable equation as follows:

Step 1. Rearrange the equation into the form (8.6) and turn it into the differential form

$$h(y)dy = g(x)dx.$$

Step 2. Integrate both sides of the equation in Step 1 (with respect to y on the left and with respect to x on the right), i.e.,

$$\int h(y)dy = \int g(x)dx.$$

Step 3. If $H(y)$ and $G(x)$ are some antiderivatives of $h(y)$ and $g(x)$ respectively, then

$$H(y) = G(x) + c$$

defines a family of solutions implicitly. It may be possible to solve this equation explicitly for y.

Example 8.5 Solve the differential equation

$$\frac{dy}{dx} = -4xy^2 \tag{8.8}$$

and then solve the initial-value problem

$$\frac{dy}{dx} = -4xy^2, \quad \underline{y(0) = 1.}$$

จาก $\frac{dy}{dx} = -4xy^2$

Separable form

$$\frac{1}{y^2}\frac{dy}{dx} = -4x$$

Differential form

$$\frac{1}{y^2}dy = -4x\,dx$$

อินทิเกรต

$$\int \frac{1}{y^2}dy = \int -4x\,dx$$

$$-\frac{1}{y} = -\frac{4x^2}{2} + C'$$

$$\frac{1}{y} = 2x^2 + C$$ เมื่อ $C = -C'$

$$\boxed{y = \frac{1}{2x^2+C}}$$

เมื่อ $y(0) = 1$

$$1 = \frac{1}{2(0)+C}$$

$$\boxed{C = 1}$$

$$\therefore \boxed{y = \frac{1}{2x^2+1}}$$ เมื่อ $y(0)=1$

① เมื่อมีสมการ DE มา แบบ<u>แยกตัวแปร</u>

$\boxed{1}$ เขียนในรูป separable form

$$h(y)\frac{dy}{dx} = g(x) \quad \text{——} \; ♥$$

$\boxed{2}$ เขียนสมการ ♥ ในรูป differential

$$h(y)\,dy = g(x)\,dx$$

$\boxed{3}$ อินทิเกรตทั้งสองข้าง

$$\int h(y)\,dy = \int g(x)\,dx$$

<u>ข้อควรระวัง</u> อินทิเกรตเสร็จ บวก C <u>ทันที</u>

Problem 8.3 Solve the initial-value problem

$$(4y - \cos y)\frac{dy}{dx} - 3x^2 = 0, \quad y(0) = 0. \quad — ♥$$

Solution. Separable form $(4y - \cos y)\frac{dy}{dx} = 3x^2$

Differential $(4y - \cos y)dy = 3x^2 dx$

integrate ทั้งสองข้าง

$$\int (4y - \cos y)dy = \int 3x^2 dx$$

$$\frac{4y^2}{2} - \sin y = \frac{3x^3}{3} + C$$

คำตอบเป็น implicit function → $\boxed{2y^2 - \sin y = x^3 + C}$ — ★

OK

ถ้าจะเช็ค ★ เป็นคำตอบของ ♥ คือ เราต้องหาได้ $\frac{dy}{dx}$ ของ ★

∴ เราก็จะใช้ implicit diff เพื่อหา $\frac{dy}{dx}$ แล้วแทน $\frac{dy}{dx}$ ใน ♥

CHECK $4y\frac{dy}{dx} - \cos y\frac{dy}{dx} = 3x^2$

$$(4y - \cos y)\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3x^2}{4y - \cos y} \quad — ✿$$

แทน ✿ ใน ♥ $\cancel{(4y - \cos y)}\left(\frac{3x^2}{\cancel{4y - \cos y}}\right) - 3x^2 = 0$

∴ สมการเป็นจริง.

คำตอบเฉพาะ $y(0) = 0$ $2(0)^2 - \sin(0) = 0^3 + C$

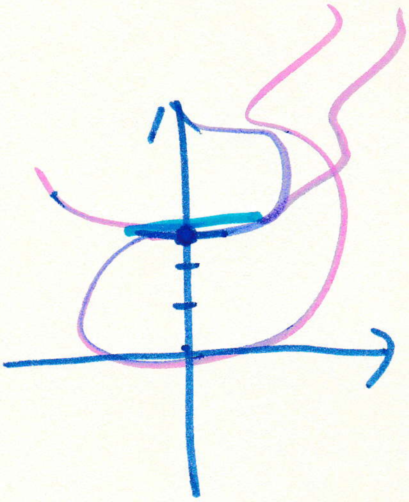
$$\therefore C = 0$$

∴ คำตอบเฉพาะ เมื่อ $y(0) = 0$ คือ $\boxed{2y^2 - \sin y = x^3.}$

Problem 8.4 Find a curve in the xy-plane that passes through $(0,3)$ and whose tangent line at a point (x,y) has slope $\frac{2x}{y^2}$.

Solution.

$$\frac{dy}{dx} = \frac{2x}{y^2} \quad , y(0)=3$$



$$y^2\frac{dy}{dx} = 2x$$

$$y^2dy = 2x\,dx$$

$$\int y^2dy = \int 2x\,dx$$

$$\frac{y^3}{3} = \frac{\cancel{2}x^2}{\cancel{2}} + C$$

$$\boxed{y^3 = 3x^2 + C}$$ ← คำตอบทั่วไป

หา $\underline{\underline{C}}$ $\quad (3)^3 = 3(0) + C$

$$C = 27.$$

∴ คำตอบเฉพาะ คือ $\boxed{y^3 = 3x^2+27}$

แบบทดสอบย่อย

เพื่อเช็คชื่อเข้าชั้นเรียน ประจำวันจันทร์ที่ 20 พฤศจิกายน พ.ศ.2560

ชื่อ-สกุล เฉลย ~ รหัสนักศึกษา.............................. ลำดับที่...........

1. จงตรวจสอบว่า ฟังก์ชัน $y = xe^x$ เป็นผลเฉลยของสมการเชิงอนุพันธ์ $y'' - y = e^x$ หรือไม่

2. จงหาผลเฉลยของปัญหาค่าเริ่มต้น

$$y' - 2x(1+y^2) = 0, y(2) = 1$$

$$\frac{dy}{dx} - 2x(1+y^2) = 0$$

$$\frac{dy}{dx} = 2x(1+y^2)$$

$$\frac{1}{1+y^2}\frac{dy}{dx} = 2x$$

$$\frac{1}{1+y^2}dy = 2x\,dx$$

$$\int \frac{1}{1+y^2}dy = \int 2x\,dx$$

$$\arctan y = \frac{2x^2}{2} + C$$

$$\boxed{\arctan y = x^2 + c}$$

คำตอบเฉพาะ

$$\arctan 1 = 4 + C$$

$$\frac{\pi}{4} = 4 + C$$

$$C = \frac{\pi}{4} - 4$$

$$\therefore \boxed{\arctan y = x^2 + \frac{\pi}{4} - 4}$$

8.2 First-order Linear Equations สมการเชิงเส้น

First-order linear equations are differential equations that can be expressed in the form

$$\frac{dy}{dx} + p(x)y = q(x). \tag{8.10}$$

Generally, we look for a general solution on some common interval where $p(x)$ and $q(x)$ are both continuous.

Notice from (8.10) that linear equations have $\frac{dy}{dx}$ and y both of degree one. If a first-order equation has this property, then it can be rewritten as (8.10).

Example 8.6

	Original equation	Linear form	$p(x)$	$q(x)$
①	$\frac{dy}{dx} + x^2y = e^x$	$\frac{dy}{dx} + x^2y = e^x$	x^2	e^x
②	$y' - 3e^xy = e^x$	$y'-3e^xy=e^x$	$-3e^x$	e^x
③	$x\frac{dy}{dx} = 1$	$\frac{dy}{dx} = \frac{1}{x}$	0	$\frac{1}{x}$
④	$xy' + 2y = 4x^2$	$y'+\frac{2}{x}y=4x$	$\frac{2}{x}$	$4x$
⑤	$\frac{dy}{dx} + y - \frac{1}{1+e^x} = 0$	$\frac{dy}{dx}+y=\frac{1}{1+e^x}$	1	$\frac{1}{1+e^x}$

☐

① $\underline{\frac{dy}{dx}} + \underbrace{x^2}_{p(x)} y = \underbrace{e^x}_{q(x)}$

② $\boxed{y' - 3e^x y = e^x} \longrightarrow \frac{dy}{dx} + \overset{p(x)}{\underline{\heartsuit}} y = \overset{q(x)}{\underline{✿}}$

$p(x) = -3e^x$

$q(x) = e^x$

③ $x\frac{dy}{dx} = 1 \longrightarrow \frac{dy}{dx} + 0 \cdot y = \frac{1}{x}$

* จัดรูปของสมการ Linear equation

สัมประสิทธิ์หน้า $\frac{dy}{dx}$ ต้องเป็น 1.

④ $xy' + 2y = 4x^2$

$\hookrightarrow y' + \underbrace{\frac{2}{x}}_{p(x)} y = \underbrace{4x}_{q(x)}$

⑤ $\frac{dy}{dx} + y - \frac{1}{1+e^x} = 0$

$\hookrightarrow \frac{dy}{dx} + y = \frac{1}{1+e^x}$

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① เลือกว่าเป็นสมการแบบไหน (Separable / เชิงเส้น)

สมมติว่า สมการเชิงเส้นอยู่ในรูป ★

$$\boxed{\frac{dy}{dx} + p(x)y = q(x)} \text{ — } \heartsuit$$

พยายาม จัดให้ ★ เป็น

$$\boxed{\frac{d}{dx}\heartsuit = \overset{q(x)}{✿}} \text{ — } ☾$$

$$d\heartsuit = ✿\,dx$$

$$\int d\heartsuit = \int ✿\,dx$$

$$\heartsuit = \int ✿\,dx + c$$

จะทำ หาอะไรมาคูณ ♥ แล้วให้เป็น ☾

ขอกำหนด $\mu(x) = e^{\int p(x)dx}$ ตัวประกอบการอินทิเกรต Integrating factor.

คูณ $\mu(x)$ ทั้งสมการ ♥

$$\boxed{\mu(x)\frac{dy}{dx} + \mu(x)p(x)y} = \mu(x)q(x)$$

$$\frac{d}{dx}(\mu(x)y) = \underline{\mu(x)q(x)}$$

$$d(\mu(x)y) = \mu(x)q(x)dx$$

$$\int d(\mu(x)y) = \int \mu(x)q(x)dx$$

$$\mu(x)y = \int \mu(x)q(x)dx + C$$

DEFINITION Given a linear equation (8.10), an **integrating factor** for (8.10) is defined by

$$\mu = e^{P(x)}$$

for some antiderivative $P(x)$ of $p(x)$.

Remark. A linear equation has infinitely many integrating factors, due to the constant of integration from $\int p(x)dx$. So, different integrating factors only differ by their coefficients. Therefore, a general solution obtained by this method is independent of the choice of integrating factors. We may write

$$\mu = e^{\int p(x)dx}$$

Below is a summary of how to apply the Method of Integrating Factors to the equation (8.10). **If the original equation is not in the form of (8.10), it needs to be rewritten first!**

Step 1. Calculate $\int p(x)dx$ and choose an integrating factor

$$\mu = e^{P(x)} = e^{\int p(x)dx}$$

for the linear equation. You may take the constant of integration to be zero to make things simpler.

Step 2. Multiply both sides of (8.10) by μ and express the result as

$$\frac{d}{dx}(\mu y) = \mu q(x).$$

Step 3. Integrate both sides of the result from Step 2 and then solve for y.