แบบฝึกหัดระคน เรื่องการหาปริพันธ์ (แบบรวมเทคนิค)

ประกอบการเรียนเพิ่มเติม วันจันทร์ที่ 13 พฤศจิกายน พ.ศ.2560

1. ปริพันธ์สามคล้าย (ดูเหมือนคล้ายกัน แต่ทำวิธีต่างกัน)

$$\int \frac{1}{1+x^2}dx \quad \text{vs} \quad \int \frac{x}{1+x^2}dx \quad \text{vs} \quad \int \frac{x}{(1+x^2)^3}dx$$

✔ ✔ ✔

2. ปริพันธ์ที่ต้องจัดรูปให้อยู่ในรูปแบบที่คุ้นเคย หรือเลือกแทนค่าก่อน แล้วใช้เทคนิคต่างๆ มาช่วย

✔ (a) $\displaystyle\int \frac{e^x}{e^{2x}+2e^x+5}dx$

Additional :

$$\int \frac{(1+\sec^2\theta)\sec^2\theta}{1+\tan^3\theta}\, d\theta$$

✔ (b) $\displaystyle\int \frac{\cos\theta}{\sin^2\theta - 6\sin\theta + 12}d\theta$

✔ (c) $\displaystyle\int e^{\sqrt{3x+9}}dx$

(d) $\displaystyle\int x^3 e^{x^2}dx$

(e) $\displaystyle\int \frac{\sin\theta}{\cos^3\theta + \cos\theta}d\theta$

✔ (f) $\displaystyle\int e^{2x}\sqrt{e^{2x}-1}dx$

3. ปริพันธ์ที่ต้องลองจัดรูปทางพีชคณิตดูก่อน แล้วค่อยใช้สูตร หรือเทคนิคการอินทิเกรตต่างๆ มาช่วย

✔ (a) $\displaystyle\int \frac{x}{\sqrt{6x-x^2}}dx$

(b) $\displaystyle\int \frac{x+4}{\sqrt{x^2+6x+10}}dx$

(c) $\displaystyle\int \frac{x+3}{\sqrt{x^2+2x+2}}dx$

4. By parts เลือกเหมาะๆ

(a) $\displaystyle\int \frac{xe^x}{(1+x)^2}dx$

5. หนึ่งปริพันธ์อาจทำได้หลายแบบ

(a) $\displaystyle\int \sqrt{1-x^2}dx$

(b) $\displaystyle\int \sec^{-1}(\sqrt{\theta})d\theta$

(c) $\displaystyle\int x^3(\sqrt{4-x^2})dx$

Ex ① • $\int \frac{1}{1+x^2}dx = \arctan(x) + C$ #

• $\int \frac{x}{1+x^2}dx$ $\quad u = 1+x^2$, $du = 2x\,dx$

$= \frac{1}{2}\int \frac{du}{u} = \ln|u| + C = \frac{1}{2}\ln|1+x^2| + C$ # Ⓐ

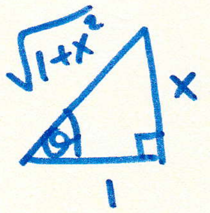
• $\int \frac{x}{(1+x^2)^3}dx$ $\quad u = 1+x^2$, $du = 2x\,dx$

$= \frac{1}{2}\int \frac{du}{u^3} = \frac{1}{2}\int u^{-3}du = \frac{1}{2}\cdot\frac{u^{-2}}{(-2)} + C$

$= -\frac{1}{4}\cdot\frac{1}{(1+x^2)^2} + C$ #

• $\int \frac{x}{1+x^2}dx$ แทนค่าตรีโกณ $\quad x = \tan\theta$, $dx = \sec^2\theta\, d\theta$

a u



$\int \frac{\tan\theta}{1+\tan^2\theta}\sec^2\theta\, d\theta = \int \frac{\tan\theta}{\cancel{\sec^2\theta}}\cancel{\sec^2\theta}\, d\theta$

$= \int \tan\theta\, d\theta$

$= \ln|\sec\theta| + C$

$= \ln|\sqrt{1+x^2}| + C$ # Ⓑ

$= \ln|(1+x)^{\frac{1}{2}} + C$

$= \frac{1}{2}\ln|1+x^2| + C$ #

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② $\boxed{a.}$ $\int \frac{e^x}{e^{2x}+2e^x+5}dx$ $\quad$ แทน $u = e^x$; $du = e^x dx$

$= \int \frac{e^x\,dx}{u^2+2u+5} = \int \frac{du}{u^2+2u+5}$

$= \int \frac{du}{(u^2+2u+1)-1+5}$

$= \int \frac{du}{(u+1)^2+4}$

ให้ $v = u+1$; $dv = du$

$= \int \frac{dv}{v^2+4}$

$= \frac{1}{2}\arctan\left(\frac{v}{2}\right) + C$

$= \frac{1}{2}\arctan\left(\frac{u+1}{2}\right) + C$

$= \frac{1}{2}\arctan\left(\frac{e^x+1}{2}\right) + C$ #

$\boxed{b}$ $\displaystyle\int \frac{\cos\theta}{\sin^2\theta - 6\sin\theta + 12}\,d\theta$ ให้ $u = \sin\theta$; $du = \cos\theta\,d\theta$

$$= \int \frac{du}{u^2 - 6u + 12}$$

$$= \int \frac{du}{(u^2 - 2u(3) + 9) + 3}$$

$$= \int \frac{du}{(u-3)^2 + 3}$$

$$= \frac{1}{\sqrt{3}}\arctan\left(\frac{u-3}{\sqrt{3}}\right) + C$$

$$= \frac{1}{\sqrt{3}}\arctan\left(\frac{\sin\theta - 3}{\sqrt{3}}\right) + C \;\#$$

เปลี่ยนโจทย์

$$\int \frac{\cos\theta}{\sin^2\theta - 7\sin\theta + 12}\,d\theta$$

$$= \int \frac{du}{u^2 - 7u + 12}$$

$$= \int \frac{1}{(u-3)(u-4)}\,du$$

พิจารณา $\dfrac{1}{(u-3)(u-4)} = \dfrac{A}{u-3} + \dfrac{B}{u-4}$

$$1 = A(u-4) + B(u-3)$$

$$1 = \underline{Au} - 4A + \underline{Bu} - 3B$$

$$1 = (A+B)u + (-4A - 3B)$$

$A + B = 0 \Rightarrow A = -B$

$-4A - 3B = 1$

$4B - 3B = 1$

$\therefore \boxed{B = 1} \Rightarrow A = -1$

$$\int \frac{1}{(u-3)(u-4)}\,du = \int \left(-\frac{1}{u-3} + \frac{1}{u-4}\right) du$$

$$= -\ln|u-3| + \ln|u-4| + C$$

$$= -\ln|\sin\theta - 3| + \ln|\sin\theta - 4| + C \;\#$$

$$\int \frac{du}{u^2 - 7u + 12}$$

$$= \int \frac{du}{\left(u^2 - 2u\cdot\frac{7}{2} + \frac{7}{2}\right)^2 - \left(\frac{7}{2}\right)^2 + 12}$$

$$= \int \frac{du}{\left(u - \sqrt{\frac{7}{2}}\right)^2 - \frac{49}{4} + 12}$$

$$= \int \frac{du}{\left(u - \sqrt{\frac{7}{2}}\right)^2 - \frac{1}{4}}$$

แทนค่าตรีโกณ

$$u - \sqrt{\frac{7}{2}} = \frac{1}{2}\sec\theta$$

$\vdots$

ตัวใครตัวมัน 😛

C $\int e^{\sqrt{3x+9}}\,dx$

By Parts

$u = 3x+9$	$u = e^{\sqrt{3x+9}}$; $dv = dx$
$du = 3dx$	$du = e^{\sqrt{3x+9}}\dfrac{3}{2\sqrt{3x+9}}dx$; $v = x$
$= \frac{1}{3}\int e^{\sqrt{3x+9}}\,dx$	$\int e^{\sqrt{3x+9}}dx = xe^{\sqrt{3x+9}} - \frac{3}{2}\int \dfrac{x}{\sqrt{3x+9}}e^{\sqrt{3x+9}}dx$
$= \frac{1}{3}\int e^{\sqrt{u}}\,du$?!?	⋮

ให้ $m = \sqrt{3x+9}$ $\quad dm = \dfrac{3}{2\sqrt{3x+9}}dx$ $\Rightarrow dx = \frac{2}{3}m\,dm$

m

$$\therefore \int e^{\sqrt{3x+9}}dx = \int \tfrac{2}{3}e^m m\,dm$$

$$= \tfrac{2}{3}\boxed{\int me^m dm}$$

by parts

$u = m$; $dv = e^m dm$

$du = dm$; $v = e^m$

$$= \tfrac{2}{3}\left[me^m - \int e^m dm\right]$$

$$= \tfrac{2}{3}\left[me^m - e^m\right] + C$$

$$= \tfrac{2}{3}\sqrt{3x+9}\,e^{\sqrt{3x+9}} - \tfrac{2}{3}e^{\sqrt{3x+9}} + C \quad \#$$

การบวกเข้า-ลบออก

3 b $\displaystyle\int \frac{x}{\sqrt{x^2+6x+10}}dx$

③ $\boxed{\theta}$ $\int \frac{x}{\sqrt{6x-x^2}}\,dx$

- $\int \frac{x}{\sqrt{6x-x^2}}\,dx = \int \frac{x}{\sqrt{x(6-x)}}\,dx$?!?

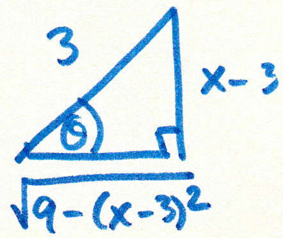
①. $\int \frac{x}{\sqrt{-(x^2-6x)}}\,dx = \int \frac{x}{\sqrt{-(x^2-6x+9)+9}}\,dx$

$= \int \frac{x}{\sqrt{9-(x-3)^2}}\,dx$

แทนค่าตรีโกณ $x-3 = 3\sin\theta \Rightarrow x = 3+3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$= \int \frac{(3+3\sin\theta)}{\sqrt{9-9\sin^2\theta}} \cdot 3\cos\theta\,d\theta$



$= \int \frac{(3+3\sin\theta)}{\cancel{3\cos\theta}}\,\cancel{3\cos\theta}\,d\theta$

$= 3\int (1+\sin\theta)\,d\theta$ $\left(\sin\theta = \frac{x-3}{3}\right)$

$= 3\theta + 3(-\cos\theta) + C$

$= 3\arcsin\left(\frac{x-3}{3}\right) - \frac{3\sqrt{6x-x^2}}{3} + C$

$= 3\arcsin\left(\frac{x-3}{3}\right) - \sqrt{6x-x^2} + C$ #

② แทน $u = 6x-x^2$; $du = (6-2x)\,dx$

$\Rightarrow 6-2x$

$\int \frac{x}{\sqrt{6x-x^2}}\,dx = \int \frac{6}{\sqrt{6x-x^2}}\,dx - \int \frac{2x}{\sqrt{6x-x^2}}\,dx - \int \frac{6}{\sqrt{6x-x^2}}\,dx + \int \frac{3x}{\sqrt{6x-x^2}}\,dx$

$= \int \frac{6-2x}{\sqrt{6x-x^2}}\,dx - \int \frac{(6-3x)}{\sqrt{6x-x^2}}\,dx$

↓

$\int \frac{x}{\sqrt{6x-x^2}}\,dx = -\frac{1}{2}\int \frac{6-2x}{\sqrt{6x-x^2}}\,dx + \int \frac{3}{\sqrt{6x-x^2}}\,dx$

$\boxed{\text{ลูกรัก}} = (6-2x)\,dx$

บวกเข้า-ลบออก

$$\int \frac{x}{\sqrt{6x-x^2}}\,dx = -\frac{1}{2}\int \frac{6-2x}{\sqrt{6x-x^2}}\,dx + \int \frac{3}{\sqrt{6x-x^2}}\,dx$$

$$= \boxed{-\frac{1}{2}\int \frac{6-2x}{\sqrt{6x-x^2}}\,dx} + \boxed{\int \frac{3}{\sqrt{6x-x^2}}\,dx}$$

$u = 6x - x^2$

$du = (6-2x)\,dx$

$$-\frac{1}{2}\int \frac{6-2x}{\sqrt{6x-x^2}}\,dx = -\frac{1}{2}\int \frac{du}{\sqrt{u}}$$

$$= -\frac{2}{2}\,\frac{u^{\frac{1}{2}}}{1}$$

$$= -\sqrt{u}$$

$$= -\sqrt{6x-x^2}$$

(เดี๋ยวต่อบวก C)

$$\int \frac{3}{\sqrt{6x-x^2}}\,dx = 3\int \frac{1}{\sqrt{9-(x-3)^2}}\,dx$$

$$= 3\arcsin\left(\frac{x-3}{3}\right)$$

(เดี๋ยวต่อบวก C

$$= -\sqrt{6x-x^2} + 3\arcsin\left(\frac{x-3}{3}\right) + C \quad \#$$

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② f $\quad \int e^{2x}\sqrt{e^{2x}-1}\,dx$

ให้ $u = e^{2x} - 1$

$du = 2e^{2x}\,dx$

$$\therefore \int e^{2x}\sqrt{e^{2x}-1}\,dx = \frac{1}{2}\int \sqrt{u}\,du$$

$$= \frac{\cancel{2}}{\cancel{2}}\,\frac{u^{\frac{3}{2}}}{3} + C$$

$$= \frac{(e^{2x}-1)^{3/2}}{3} + C$$

เปลี่ยนโจทย์

$$\int e^{x}\sqrt{e^{2x}-1}\,dx$$

ให้ $u = e^{x}$; $du = e^{x}\,dx$

$$= \int \sqrt{u^2-1}\,du$$

ให้ $u = \sec\theta$

$du = \sec\theta\tan\theta\,d\theta$

$$\therefore \int \sqrt{u^2-1}\,du = \int \sqrt{\sec^2\theta - 1}\,\sec\theta\tan\theta\,d\theta$$

$$= \int \tan\theta\sec\theta\tan\theta\,d\theta$$

$$= \boxed{\int \tan^2\theta\sec\theta\,d\theta}$$

$$= \int (\sec^2\theta - 1)\sec\theta\,d\theta$$

$= \int (\sec^3\theta - \sec\theta)\, d\theta$

$= \int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta$ ✓

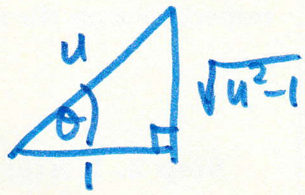
ฝึกทำ $\int \sec^3\theta\, d\theta$

Hint by parts แนวๆ

$u = \sec\theta \quad ; \quad dv = \sec^2\theta\, d\theta$

$du = \sec\theta\tan\theta\, d\theta \quad ; \quad v = \tan\theta$

$$\boxed{\int \sec^3\theta\, d\theta = \sec\theta\tan\theta - \int \sec\theta\tan^2\theta\, d\theta}$$



$$\int \tan^2\theta\sec\theta\, d\theta = \int \sec^3\theta\, d\theta - \int \sec\theta\, d\theta$$

$$\int \tan^2\theta\sec\theta\, d\theta = \sec\theta\tan\theta - \int \tan^2\theta\sec\theta\, d\theta - \int \sec\theta\, d\theta$$

$$2\int \tan^2\theta\sec\theta\, d\theta = \sec\theta\tan\theta - \ln|\sec\theta + \tan\theta| + C$$

$$\int \tan^2\theta\sec\theta\, d\theta = \frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\ln|\sec\theta + \tan\theta| + C$$

$\shortparallel$

$$\int \sqrt{u^2-1}\, du = \frac{1}{2}u\sqrt{u^2-1} - \frac{1}{2}\ln|u + \sqrt{u^2-1}| + C$$

$$= \frac{1}{2}e^x\sqrt{e^{2x}-1} - \frac{1}{2}\ln|e^x + \sqrt{e^{2x}-1}| + C \quad \#$$