206331 Advanced Calculus.

CH 1 Differential Calculus of multivariable functions.

Notation $\mathbb{R}$: set of real number

Relation : $A, B \to A \times B$

$$A \times B = \{(a,b) \mid a \in A \text{ and } b \in B\}.$$

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{ \underbrace{(a,b)}_{\text{ordered pair}} \mid a \in \mathbb{R} \text{ and } b \in \mathbb{R} \}$$

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{n \text{ terms.}}$$

Q : สมาชิกใน $\mathbb{R}^n$ คืออะไร ?

$$\mathbb{R}^n = \{ \underbrace{(x_1, \ldots, x_n)}_{n\text{-tuples.}} \mid x_1, \ldots, x_n \in \mathbb{R} \}$$

$$\mathbb{R}^n \times \mathbb{R} = \{ (x_1, \ldots, x_n, z) \mid x_1, \ldots, x_n \in \mathbb{R}, z \in \mathbb{R} \}$$

ทบทวน (Function of one variable)

Def^{n} : A relation $f \subseteq A \times B$ is a function / mapping if and only if

For each $(x_1, y_1), (x_2, y_2) \in f$

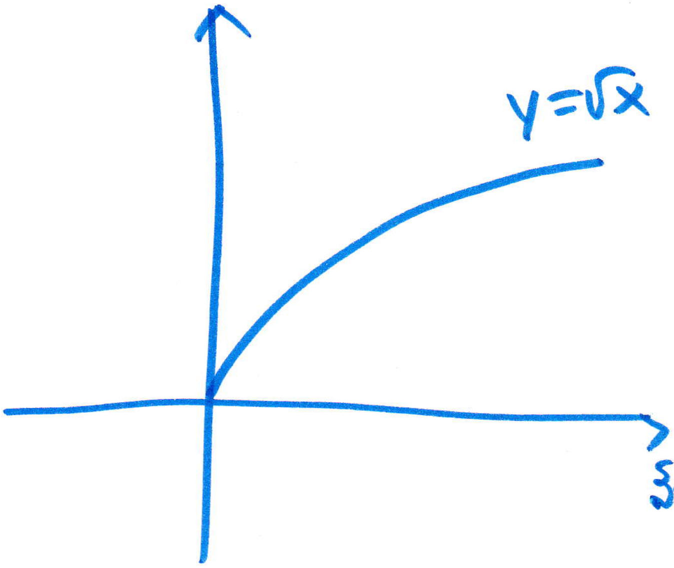
if $x_1 = x_2$, then $y_1 = y_2$

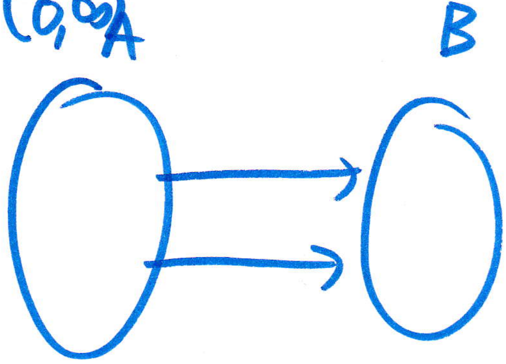
$\to$ $\text{Dom}(f) = D_f = \{ x \in A \mid \exists y \in B ; (x,y) \in f \}$

$\to$ $\text{Range}(f) = R_f = \{ y \in B \mid \exists x \in B ; (x,y) \in f \}$

Ex $f(x)=\sqrt{x}$

$D_f = [0,\infty)$
$R_f = [0,\infty)$





~~HW~~ ฝึก พิสูจน์ว่า

สำหรับ $f(x)=\sqrt{x}$

$\boxed{D_f = [0,\infty)}$, $R_f = [0,\infty)$

① จะแสดงว่า $D_f = [0,\infty)$

($\subseteq$) แสดงว่า $D_f \subseteq [0,\infty)$

ให้ $x \in D_f$

⋮

$x \in [0,\infty)$

($\supseteq$) แสดงว่า $[0,\infty) \subseteq D_f$

ให้ $x \in [0,\infty)$

⋮

$x \in D_f$.

② จะแสดงว่า $R_f = [0,\infty)$

($\subseteq$)

($\supseteq$)

How can we generalize the definition of function to the higher dimension?

(n+1)-tuples

Def$^{\underline{n}}$ Let $f \subseteq \mathbb{R}^n \times \mathbb{R}$ be a relation. (if and only if)

f is a function of $\underline{n}$ variables. iff

for each $(P_1, W_1), (P_2, W_2) \in f$ s.t.

$(p_1^1, \ldots, p_n^1, w_1)$, $(p_1^2, \ldots, p_n^2, w_1)$

$P_1, P_2 \in \mathbb{R}^n$, $W_1, W_2 \in \mathbb{R}$

if $P_1 = P_2$, then $W_1 = W_2$

Domain of f : $D_f = \{P \in \mathbb{R}^n \mid \exists W \in \mathbb{R};\ (P, W) \in f\}$.

Range of f : $R_f = \{W \in \mathbb{R} \mid \exists P \in \mathbb{R}^n;\ (P, W) \in f\}$.

- f is well-defined. $\Leftrightarrow$ f is a function.

Rmk ① $z = f(x, y)$: x, y are independent variables. (ตัวแปรอิสระ / ตัวแปรต้น)

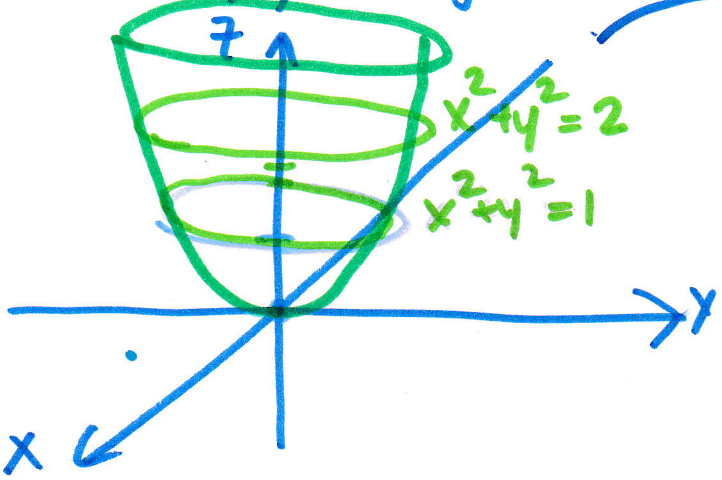
z is dependent variable. (ตัวแปรตาม)

↓

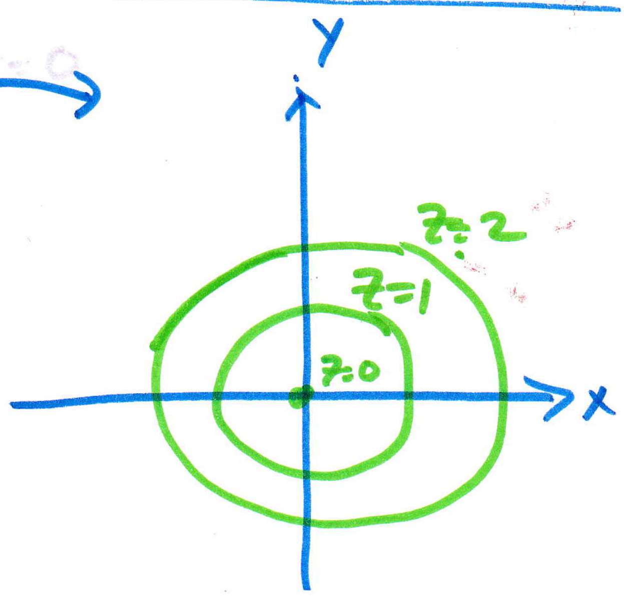
ฟังก์ชัน 2 ตัวแปร

Graphical Representation

Ex $z = f(x, y) = x^2 + y^2$



contour map / level surface



Find the domain of functions

(1) $f(x,y) = 4x^2+4y^2+z^2$

(2) $f(x,y) = \sqrt{x^2-y^2}$

(3) $f(x,y) = \sqrt{x^2+y^2-1} + \ln(4-x^2-y^2)$

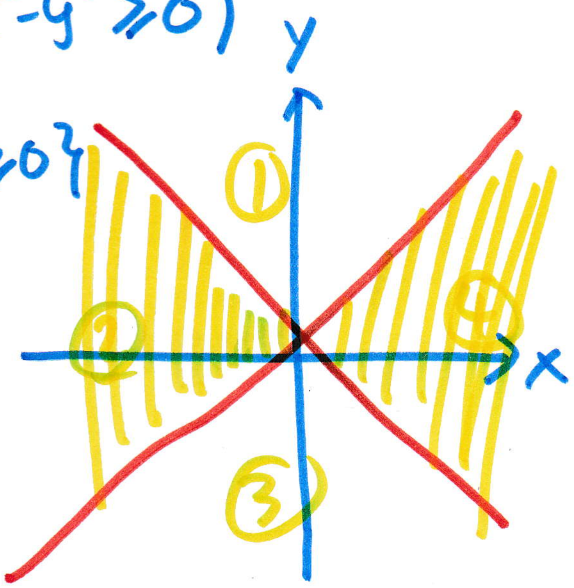
---

(1) $f(x,y) = 4x^2+4y^2+z^2$ $\quad D_f = \{(x,y)\in\mathbb{R}^2 \mid \exists w\in\mathbb{R}; (x,y,w)\in$

$\therefore D_f = \mathbb{R}^2$

(2) $f(x,y) = \sqrt{x^2-y^2}$: (ทด $x^2-y^2 \geq 0$)

$D_f = \{(x,y)\in\mathbb{R}^2 \mid x^2-y^2\geq 0\}$



วาด อาณาบริเวณของโดเมน

พดรภด $x^2-y^2\geq 0$

(3) $f(x,y) = \sqrt{x^2+y^2-1} + \ln(4-x^2-y^2)$

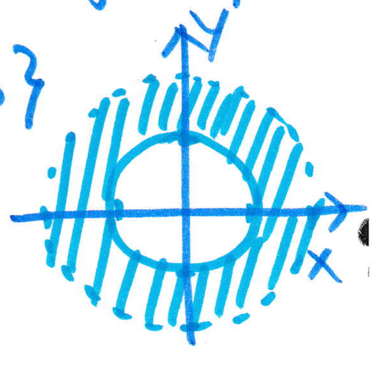
Condition 1 $\quad x^2+y^2-1 \geq 0$

Condition 2 $\quad 4-x^2-y^2 > 0$

$D_f = \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\geq 1\} \cap \{(x,y)\in\mathbb{R}^2 \mid 4-x^2-y^2>0\}$

$= \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\geq 1 \wedge 4-x^2-y^2>0\}$

$= \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2\geq 1 \wedge x^2+y^2<4\}$



Let $f = \{(x,y,z) \in \mathbb{R}^3 \mid z = \sqrt{9 - x^2 - y^2}\}$

(1) Is f well-defined?

(2) Find its domain and Range. ~~of f~~

(3) Verify its domain and range.

Proof (1) To show that f is well-defined,

ทอ $\left((P_1, W_1), (P_2, W_2) \in \mathbb{R}^2 \times \mathbb{R}, \text{ If } P_1 = P_2, \ (P_1, P_2 \in \mathbb{R}^2) \text{ then } W_1 = W_2 \right)$

Let $(P_1, W_1), (P_2, W_2) \in \mathbb{R}^2 \times \mathbb{R}$ such that $P_1 = P_2$

Since $P_1, P_2 \in \mathbb{R}^2$, assume that $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$

$P_1 = P_2$ implies $(x_1, y_1) = (x_2, y_2)$

$\therefore x_1 = x_2$ and $y_1 = y_2$

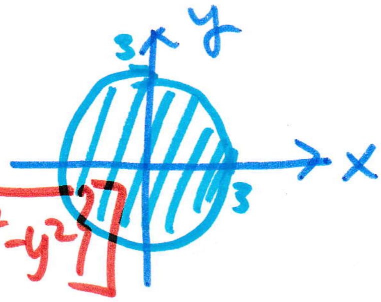
$x_1^2 = x_2^2$; $y_1^2 = y_2^2$

$\therefore 9 - x_1^2 - y_1^2 = 9 - x_2^2 - y_2^2$

if $9 - x_1^2 - y_1^2 \geq 0$, $9 - x_2^2 - y_2^2 \geq 0$;

$z_1 = \sqrt{9 - x_1^2 - y_1^2} = \sqrt{9 - x_2^2 - y_2^2} = z_2$

(2) $D_f = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 9\}$



นิยาม โดเมน $\left[D_f = \{(x,y) \in \mathbb{R}^2 \mid \exists z \in \mathbb{R} \text{ s.t. } z = \sqrt{9 - x^2 - y^2}\} \right]$

$(\subseteq)$ $\boxed{\text{Claim } D_f \subseteq \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 9\}}$

จาก $\left(\begin{array}{l} \text{Let } P \in D_f \quad (P \in \mathbb{R}^2, \\ \text{s.t. } P = (a,b) \end{array} \right)$

Let $(a,b) \in D_f$

$\therefore (a,b) \in \mathbb{R}^2, \exists c \in \mathbb{R}$ s.t. $c = \sqrt{9-a^2-b^2}$

Consider; $c = \sqrt{9-a^2-b^2}$; $c \geq 0$

$c^2 = 9 - a^2 - b^2$

$\therefore \quad a^2+b^2 = 9 - c^2 \leq 9 \quad (\because c^2 \geq 0)$

$\therefore (a,b) \in \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 9\}$

$(\supseteq)$ $\boxed{\text{Claim } \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 9\} \subseteq D_f}$

Let $(a,b) \in \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 9\}$

$[\exists c \in \mathbb{R}$ s.t. $c = f(a,b)]$ (เลือก $c \in \mathbb{R}$)

$\therefore (a,b) \in \mathbb{R}^2$ s.t. $a^2+b^2 \leq 9$

$\therefore 9-a^2-b^2 \geq 0$

~~then~~ choose $c = \sqrt{9-a^2-b^2} \in \mathbb{R}$

such that $c = f(a,b)$

$(a,b) \in D_f$

$\therefore D_f = \{(x,y) \in \mathbb{R}^2 \mid x^2+y^2 \leq 9\}.$

(3) Range of f : $R_f = [0,3]$ $\left(z = \sqrt{9-x^2-y^2}\right)$

นิยามเรนจ์ $\left[R_f = \{z \in \mathbb{R} \mid \exists (x,y) \in \mathbb{R}^2 \text{ s.t. } z = \sqrt{9-x^2-y^2}\}\right]$.

($\subseteq$) $\boxed{\text{Claim } R_f \subseteq [0,3]}$ $c \in [0,3]$

Let $c \in R_f$ $\therefore c \in \mathbb{R}$; and

there exists $(a,b) \in \mathbb{R}^2$ s.t. $c = \sqrt{9-a^2-b^2}$

$\therefore c \geq 0$

$\therefore c^2 = 9 - a^2 - b^2 \leq 9$

$\therefore c^2 \leq 9 \Rightarrow (-3 \leq c \leq 3) \cap c \geq 0$

$\Rightarrow 0 \leq c \leq 3$

$\therefore c \in [0,3]$

($\supseteq$) $\boxed{\text{Claim } [0,3] \subseteq R_f}$

อยากได้

Let $c \in [0,3]$ $\therefore$ $\left[\exists\, \underline{(a,b) \in \mathbb{R}^2} \text{ s.t. } c = \sqrt{9-a^2-b^2}\right]$

choose $(a,b) = \left(\sqrt{9-c^2}, 0\right) \in \mathbb{R}^2$ s.t.

$\left[\begin{array}{l} c \in [0,3] \\ \therefore \sqrt{9-c^2} \in \mathbb{R} \end{array}\right]$

$f(a,b) = \sqrt{9-\left(\sqrt{9-c^2}\right)^2 - 0}$

$= \sqrt{9-(9-c^2)-0}$

$= \sqrt{c^2} = c$

$\therefore c \in R_f$

$\therefore R_f = [0,3]$ #