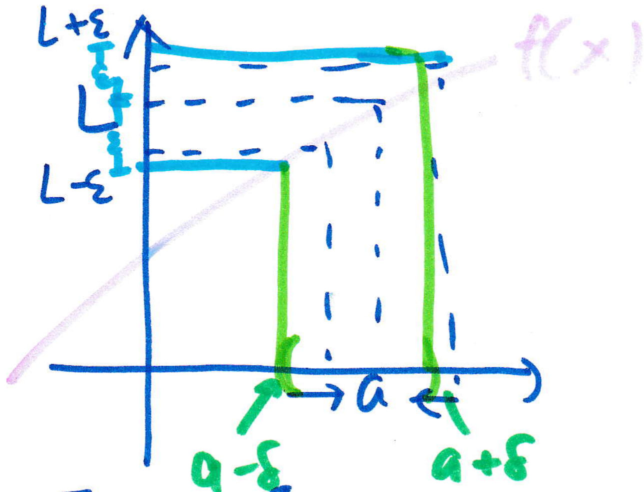
ทบทวน (Revision)

- f is well-defined.
- D_f
- R_f

Limit of function



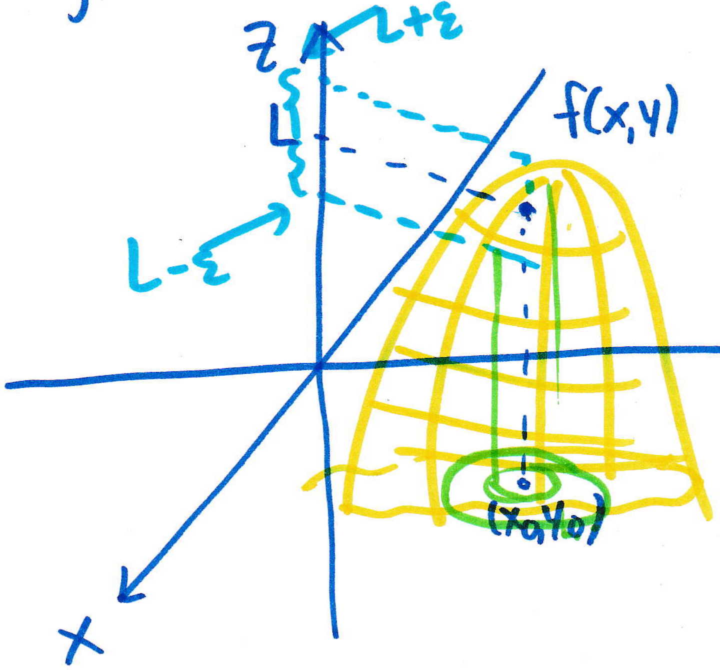
$$\lim_{x \to a} f(x) = L$$

$$\lim_{x \to a} f(x) = L \iff \forall \varepsilon > 0\ \exists \delta > 0\ [\,|f(x) - L| < \varepsilon$$
whenever $|x - a| < \delta$

$$\iff \forall \varepsilon > 0\ \exists \delta > 0\ [\,|x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon\,]$$

Q: If f is a multivariable function, how can we generalize this definition?

$z = f(x,y)$



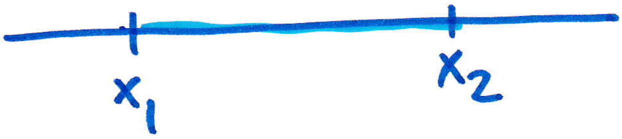
$$\boxed{\lim_{(x,y) \to (x_0,y_0)} f(x,y) = L}$$

อยากทำ อยากนิยามใกล้/ไกล ใน $\mathbb{R}^2$

Topology in $\mathbb{R}^n$

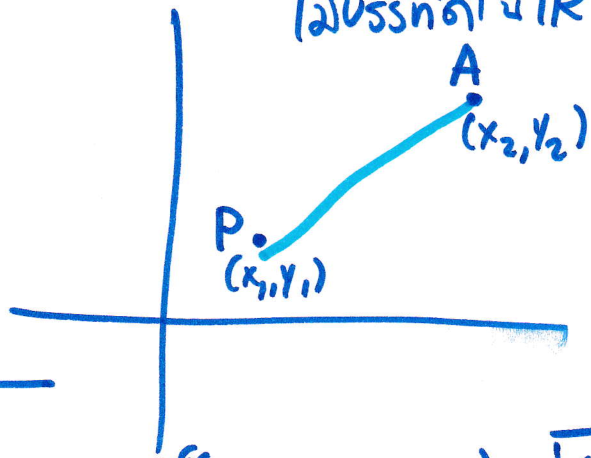
นิยามไม้บรรทัดใน $\mathbb{R}^n$

Ex ไม้บรรทัดใน $\mathbb{R}^1$



$$d(x_1,x_2) = |x_2 - x_1|$$

ไม้บรรทัดใน $\mathbb{R}^2$



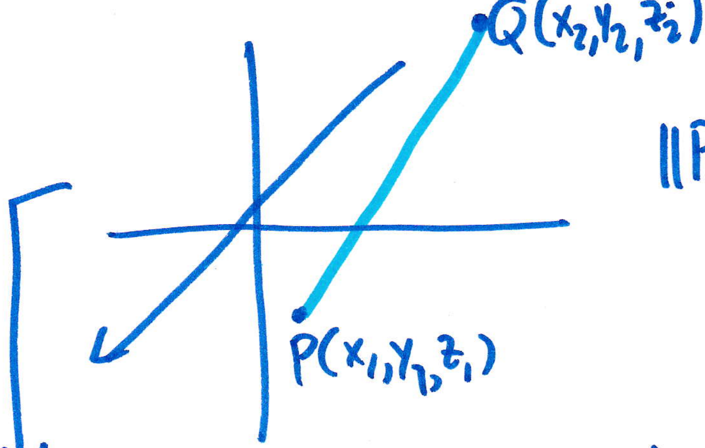
$$d((x_1,y_1),(x_2,y_2)) = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

เมตริก. Euclidean metric

$$\|P-A\| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$$

(Euclidean norm)

ไม้บรรทัดใน $\mathbb{R}^3$



$$\|P-Q\| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$$

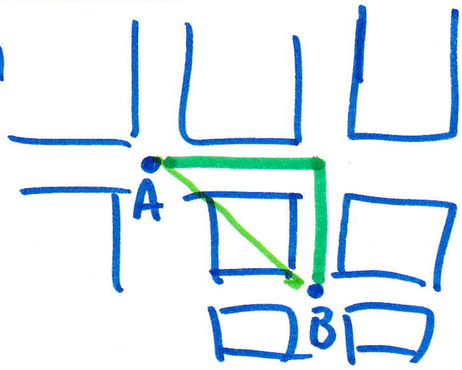
Euclidean norm of $\mathbb{R}^n$ $\vec{X} = (x_1,x_2,\dots,x_n)$ $\vec{A} = (a_1,a_2,\dots,a_n)$

$$\|P-A\| = \sqrt{(x_1-a_1)^2+(x_2-a_2)^2+\dots+(x_n-a_n)^2}$$

Norm อื่นๆ Taxicab norm

$P=(x_1,y_1)$, $Q(x_2,y_2)$

$$\|P-Q\| = |x_1-x_2|+|y_1-y_2|$$



Neighborhood (เพื่อนบ้าน)

Definition An open neighborhood (nbhd) (open ball) of $\underline{A}$ is a set of point $P \in \mathbb{R}^n$ such that

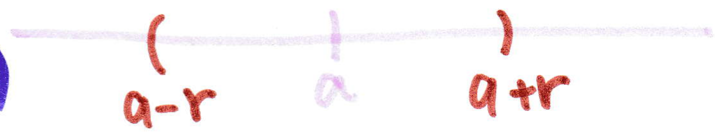
$$0 \le \|P-A\| < \underline{r} \quad \text{where } r \in \mathbb{R}^+$$

• The open nbhd. $N(A;r)$ is denoted by

$$N(A;r) = \{P \in \mathbb{R}^n \mid 0 \le \|P-A\| < r\}.$$

$\boxed{1\text{ D}}$ $\|P-A\| = |\overset{x}{P}-\overset{a}{A}| < r$

$(\mathbb{R}) \Rightarrow N(\underline{A};r) = \{x \in \mathbb{R} \mid 0 \le |x-a| < r\}$

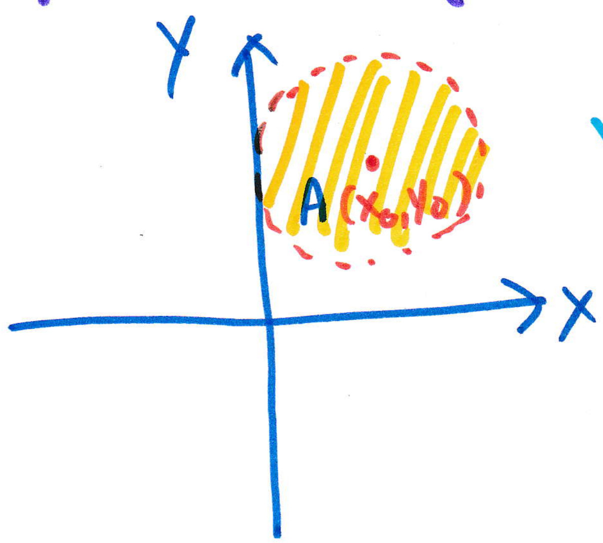


$\boxed{2\text{D}}$ $\|\overset{(x_1,y_1)}{P}-\overset{(x_0,y_0)}{A}\| = \sqrt{(x_1-x_0)^2+(y_1-y_0)^2}$

$(\mathbb{R}^2) \Rightarrow N(A;r) = \{(x,y) \in \mathbb{R}^2 \mid \underbrace{0 \le \|P-A\| < r}\}$

$$\sqrt{(x-x_0)^2+(y-y_0)^2} < r$$

$$(x-x_0)^2+(y-y_0)^2 < r^2$$



3D $\qquad \overset{(x,y,z)}{P} - \overset{(x_0,y_0,z_0)}{A}$

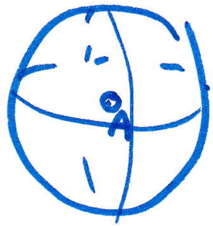
$$\|P-A\| = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$$

$\mathbb{R}^3 \Rightarrow N(A;r) = \{(x,y,z)\in\mathbb{R}^3 \mid 0 \le \|P-A\| < r\}$

interpretation : sphere radius r centered $A(x_0, y_0, z_0)$

- A closed neighborhood $N[A;r]$ is denoted by
  $N[A;r] = \{P\in\mathbb{R}^n \mid 0 \le \|P-A\| \le r\}$

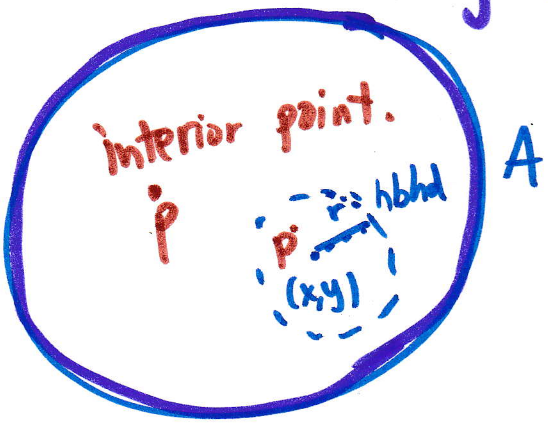
- A deleted neighborhood of a point A is a set $P\in\mathbb{R}^n$ such that $0 < \|P-A\| < r$



The deleted nbhd of A is

$$\{P\in\mathbb{R}^n \mid 0 < \|P-A\| < r\}$$

boundary of A



$P \in Nbhd(A)$

- P is an interior point
- P is not on the boundary of A.

Definition: Let f be an n-dimensional function defined on the nbhd of A.

L is the limit of f as P approaches A; written as $\lim\limits_{P \to A} f(P) = L$ iff

$\forall \varepsilon > 0 \ \boxed{\exists \delta > 0}$ s.t. $|f(P) - L| < \varepsilon$ whenever $\boxed{0 < \|P - A\| < \delta}$

There exists a deleted nbhd of A with radius δ

$$\boxed{\lim_{P \to A} f(P) = L \iff \forall \varepsilon > 0 \ \exists \delta > 0 \ [0 < \|P - A\| < \delta \Rightarrow |f(P) - L| < \varepsilon]}$$

Ex $\lim\limits_{(x,y) \to (2,1)} (x + 5y) = 7$

ทด

CLAIM. $\forall \varepsilon > 0 \ \exists \delta > 0 \ [\|(x,y) - (2,1)\| < \delta \Rightarrow |(x+5y) - 7| < \varepsilon]$

ให้ $\varepsilon > 0$

เลือก $\delta =$ ______ ยังไม่รู้...

สมมติ $\boxed{\sqrt{(x-2)^2 + (y-1)^2} < \delta}$

แสดงให้ได้ว่า $|(x+5y) - 7| < \varepsilon$ (จะแสดง)

$|x-2| = \sqrt{(x-2)^2} \le \sqrt{(x-2)^2 + (y-1)^2} < \delta$

$|y-1| = \sqrt{(y-1)^2} \le \sqrt{(x-2)^2 + (y-1)^2} < \delta$

$|(x+5y) - 7| = |(x-2) + 5y - 5|$ $< \varepsilon$

$$= |(x-2)+5(y-1)| < \varepsilon$$

Triangle inequality $|a+b| \le |a| + |b|$

$$\le |x-2| + |5(y-1)|$$

$$= \underbrace{\underset{<\delta}{|x-2|} + 5\underset{<\delta}{|y-1|}}_{\varepsilon}$$

$$< \delta + 5\delta = 6\delta = \varepsilon$$

เลือก $\delta = \frac{\varepsilon}{6}$

Proof $[\forall \varepsilon > 0\ \exists \delta > 0\ [\|(x,y)-(2,1)\| < \delta \Rightarrow |(x+5y)-7| < \varepsilon]$

Let $\varepsilon > 0$. choose $\delta = \frac{\varepsilon}{6}$ such that.

$|x-2| = \sqrt{(x-2)^2} < \sqrt{(x-2)^2+(y-1)^2} < \delta$

and $|y-1| = \sqrt{(y-1)^2} < \sqrt{(x-2)^2+(y-1)^2} < \delta$

Then $|(x+5y)-7| = |(x-2)+(5y-5)|$

$$= |(x-2)+5(y-1)|$$

$$\overset{\Delta}{\le} |x-2| + 5|y-1|$$

$$< \delta + 5\delta$$

$$= 6\delta = \cancel{6}\left(\frac{\varepsilon}{\cancel{6}}\right) = \varepsilon$$

$$\therefore \lim_{(x,y)\to(2,1)} (x+5y) = 7$$

HW (1) $\lim_{(x,y)\to(0,1)} (2x+y) = 1$ $\boxed{7}$

(2) $\lim_{(x,y)\to(1,2)} (x^2+2y) = 5$