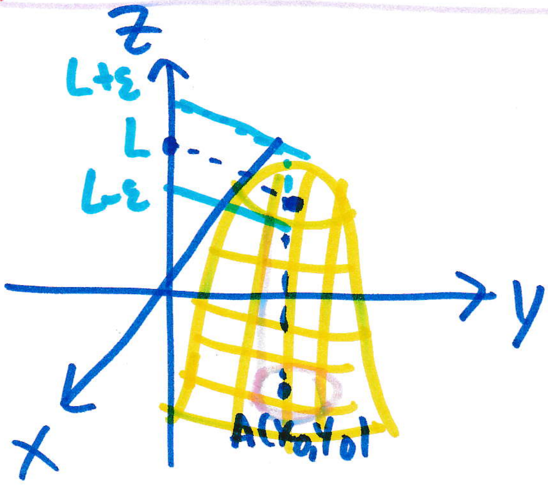
ทบทวน

- ลิมิตของฟังก์ชันสองตัวแปร. (สนใจ $f: \mathbb{R}^2 \to \mathbb{R}$)

$$\boxed{\lim_{\substack{(x,y)\to(x_0,y_0)\\ (P)\qquad (A)}} f(x,y) = L \iff \forall \varepsilon > 0\ \exists \delta > 0\ [0 < \|P - A\| < \delta \Rightarrow |f(P) - L| < \varepsilon]}$$



~~HW~~ $\displaystyle\lim_{(x,y)\to(0,1)} (2x+y) = 1$

Ex $\displaystyle\lim_{(x,y)\to(1,2)} (x^2+2y) = 5$

Claim $\displaystyle\lim_{(x,y)\to(1,2)} (x^2+2y) = 5 \iff \forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$[0 < \|(x,y) - (1,2)\| < \delta \Rightarrow |(x^2+2y) - 5| < \varepsilon]$$

ทด Let $\varepsilon > 0$

Choose $\delta = \underline{\ ?\ } > 0$

พิจารณ $0 < \sqrt{(x-1)^2 + (y-2)^2} < \delta$

$|x-1| = \sqrt{(x-1)^2} < \sqrt{(x-1)^2 + (y-2)^2} < \delta$

$|y-2| = \sqrt{(y-2)^2} < \sqrt{(x-1)^2 + (y-2)^2} < \delta$

จะได้ $|(x^2+2y) - 5| = |(x^2-1) + (2y-4)|$

$\leq |x^2 - 1| + 2|y-2| < \delta$

$$= |x+1| \cdot \underbrace{|x-1|}_{<\delta} + 2\underbrace{|y-2|}_{<\delta}$$

$$< \delta \underbrace{|x+1|}_{3} + 2\delta = 3\delta + 2\delta = 5\delta$$

→ เรารู้ $|x-1| < \delta$

เลือก $\delta = 1 \Rightarrow |x-1| < 1$

$$-1 < x-1 < 1$$

$$-3 < 1 < x+1 < 3$$

$$\therefore \; |x+1| < 3$$

เลือก $\delta = \frac{\varepsilon}{5}$ หรือ 1 ใครน้อยกว่า เลือกตัวนั้น.

$$\Rightarrow \delta = \min\left\{1, \frac{\varepsilon}{5}\right\}$$

Proof Let $\varepsilon > 0$

choose $\delta = \min\left\{1, \frac{\varepsilon}{5}\right\} > 0$

for $|x-1| = \sqrt{(x-1)^2} < \sqrt{(x-1)^2 + (y-1)^2} < \delta$

and $|y-2| = \sqrt{(y-2)^2} < \sqrt{(x-1)^2 + (y-1)^2} < \delta$

and if $|x-1| < \delta = 1$, then $|x+1| < 3$

Then $|(x^2+2y) - 5| = |(x^2-1) + (2y-4)|$

$$\overset{\Delta}{\leq} |x^2-1| + |2y-4|$$

$$= \underbrace{|x-1|}_{<3} \cdot \underbrace{|x+1|}_{<\delta} + 2\underbrace{|y-2|}_{<\delta}$$

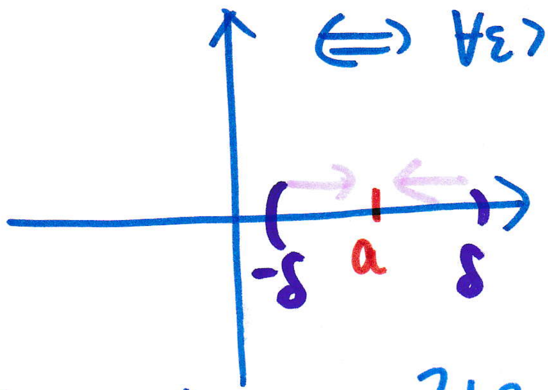
$$< 3\delta + 2\delta = 5\delta = \varepsilon \quad \#$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \iff \forall \varepsilon > 0\ \exists \delta > 0\ [0 < \|(x,y)-(x_0,y_0)\| < \delta \Rightarrow |f(x,y) - L| < \varepsilon].$$

Limit of single variable function

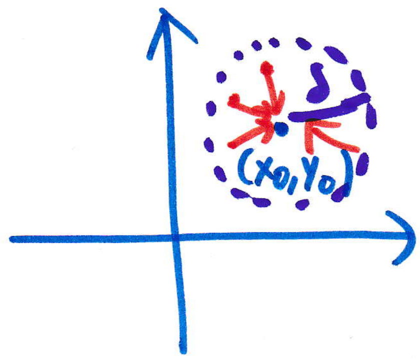
$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x)$$

$$\iff \forall \varepsilon > 0\ \exists \delta > 0\ [0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon]$$



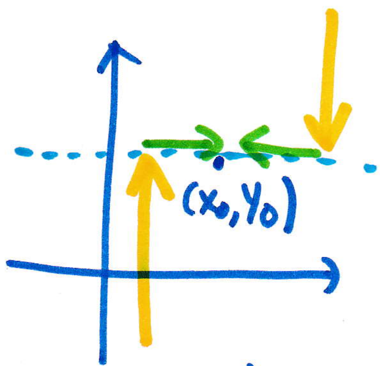
$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

?!?

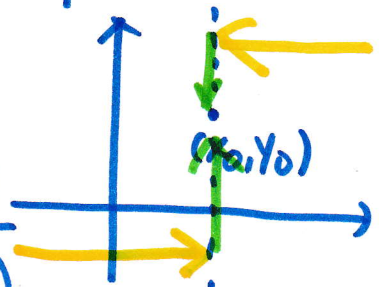


Iterated Limit (ลิมิตทำซ้ำ)

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) \quad — ①\ \lim_{x\to x_0}\left[\lim_{y\to y_0} f(x,y)\right]$$



$$— ②\ \lim_{y\to y_0}\left[\lim_{x\to x_0} f(x,y)\right]$$



Q: ถ้า $\lim_{x\to x_0}\left[\lim_{y\to y_0} f(x,y)\right] = \lim_{y\to y_0}\left[\lim_{x\to x_0} f(x,y)\right] = L$

สรุปได้ไหมว่า $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$? NO

Ex Let $f(x,y)=\dfrac{3x-2y}{x+y}$ s.t. $(x,y)\neq(0,0)$

Evaluate $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ (if exists).

Sol$^{\underline{n}}$: $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0}\dfrac{3x-2y}{x+y}\right]=\lim\limits_{x\to 0}\left[\dfrac{3x}{x}\right]=\lim\limits_{x\to 0} 3 = 3$

$\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0}\dfrac{3x-2y}{x+y}\right]=\lim\limits_{y\to 0}\left[\dfrac{-2y}{y}\right]=\lim\limits_{y\to 0}(-2) = -2$

Since $\lim\limits_{x\to 0}\left[\lim\limits_{y\to 0}\dfrac{3x-2y}{x+y}\right]=3\neq-2=\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0}\dfrac{3x-2y}{x+y}\right]$,

$\lim\limits_{(x,y)\to(0,0)} f(x,y)$ does not exist. #

Ex Let $f(x,y)=\dfrac{x^2y}{x^4+y^2}$ such that $(x,y)\neq(0,0)$

Evaluate $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ (if exists.)

Sol$^{\underline{n}}$: CHECK : $\boxed{\text{Iterated limit}}$

$\lim\limits_{y\to 0}\left[\lim\limits_{x\to 0}\dfrac{x^2y}{x^4+y^2}\right]=\lim\limits_{y\to 0}[0]=0$

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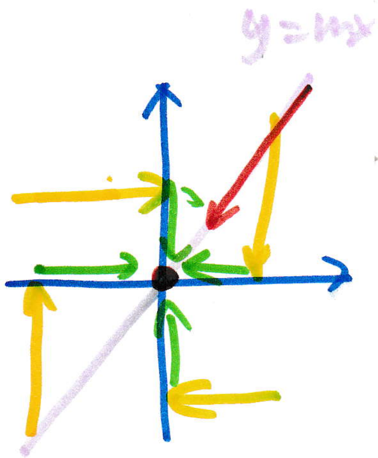


CHECK $\boxed{\text{(limit on the path)}}$

On the path $y=mx$ such that $m\neq 0$

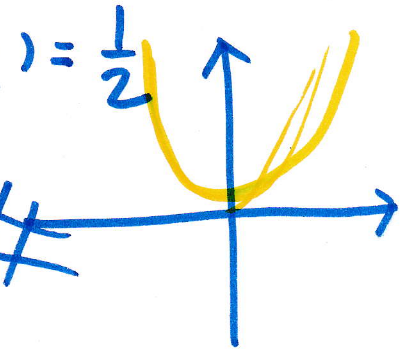
Consider $f(x,y)=f(x,mx)=\dfrac{x^2(mx)}{x^4+(mx)^2}=\dfrac{mx^3}{x^4+m^2x^2}$

$\therefore \lim\limits_{x\to 0}\left[f(x,mx)\right]=\lim\limits_{x\to 0}\dfrac{\cancel{x^2}(mx)}{\cancel{x^2}(x^2+m^2)}=\lim\limits_{x\to 0}\dfrac{mx}{x^2+m^2}=0$

On the path $y = x^2$:

$$f(x,y) = f(x,x^2) = \frac{x^2(x^2)}{x^4+(x^2)^2} = \frac{x^4}{x^4+x^4} = \frac{x^4}{2x^4}$$

$$\lim_{x\to 0}[f(x,x^2)] = \lim_{x\to 0}\frac{\cancel{x^4}}{2\cancel{x^4}} = \lim_{x\to 0}\left(\frac{1}{2}\right) = \frac{1}{2}$$



$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist. #

Ex Let $f(x,y) = \dfrac{x^2 y}{x^2+2y^2}$ s.t. $(x,y) \neq (0,0)$.

Evaluate $\lim_{(x,y)\to(0,0)} f(x,y)$ (if exists).

Sol^{n}. (1) CHECK iterated limit

$$\lim_{y\to 0}\left[\lim_{x\to 0}\frac{x^2y}{x^2+2y^2}\right] = 0$$

$$\lim_{x\to 0}\left[\lim_{y\to 0}\frac{x^2y}{x^2+2y^2}\right] = 0$$

(2) CHECK

เส้น $\boxed{y = mx}$ $(m \neq 0)$ $\lim_{x\to 0} f(x,mx) = \lim_{x\to 0}\dfrac{x^2(mx)}{x^2+2m^2x^2} = \lim_{x\to 0}\dfrac{\cancel{x^2}(mx)}{\cancel{x^2}(1+2m^2)} = 0$

$\boxed{y = mx^2}$ $(m \neq 0)$ $\lim_{x\to 0} f(x,mx^2) = \lim_{x\to 0}\dfrac{x^2(mx^2)}{x^2+2m^2x^4} = \lim_{x\to 0}\dfrac{\cancel{x^2}(mx^2)}{\cancel{x^2}(1+2m^2x^2)} = 0$

CLIAM $\lim_{(x,y)\to(0,0)} f(x,y) = 0$?!?

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0 \iff \forall \varepsilon > 0\ \exists \delta > 0\ [0 < \|(x,y)-(0,0)\| < \delta \Rightarrow |f(x,y)-0| < \varepsilon]$$

ทด Let $\varepsilon > 0$

Choose $\delta =$ ______ > 0

$$0 < \|(x,y)-(0,0)\| < \delta \Downarrow \sqrt{(x^2+y^2} < \delta$$

$$\left.\begin{matrix} |x| = \sqrt{x^2} \le \sqrt{x^2+y^2} < \delta \\ |y| = \sqrt{y^2} \le \sqrt{x^2+y^2} < \delta \end{matrix}\right.$$

$$\left|\frac{x^2y}{x^2+2y^2}\right| \underset{\text{อยากได้}}{<\varepsilon}$$

พิจารณา

$$\left|\frac{x^2y}{x^2+2y^2}\right| = \underbrace{\left|\frac{x^2}{x^2+2y^2}\right|}_{\le 1} \cdot \underbrace{|y|}_{<\delta} < \delta$$

$x^2 \le x^2+2y^2$ เลือก $\delta = \varepsilon$.

$$\Rightarrow \frac{x^2}{x^2+2y^2} \le 1$$

Proof Let $\varepsilon > 0$

choose $\delta = \varepsilon$ such that,

for $|y| = \sqrt{y^2} \le \sqrt{x^2+y^2} < \delta$,

and $x^2 \le x^2+2y^2$ implies $\dfrac{x^2}{x^2+2y^2} \le 1$,

Then $$\left|\frac{x^2y}{x^2+2y^2}\right| = \underbrace{\left|\frac{x^2}{x^2+2y^2}\right|}_{\le 1} \cdot \underbrace{|y|}_{<\delta} < \delta = \varepsilon$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2y}{x^2+2y^2} = 0. \quad \#$$

HW 1) $h(x,y) = \dfrac{x^2+y^2}{x+y}$; $(x,y) \neq (0,0)$ $\boxed{7}$

Evaluate $\lim\limits_{(x,y)\to(0,0)} h(x,y)$ (if exists)

2) $f(x,y) = \begin{cases} (x+y)\sin\left(\frac{1}{x}\right) & ; \text{if } x \neq 0 \\ 0 & ; x = 0 \end{cases}$

Evalute $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ (if exists.).