Quiz

A quiz for the class of February 9 (Fri), 2018.

Name.........KEY......................... Student ID.................. No.........

Let $f(x,y)$ be a function defined by

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$$

In the last class, we already computed the partial derivative $D_xf(x,y)$ and $D_yf(x,y)$. The results are

$$D_xf(x,y) = \begin{cases} \dfrac{2xy^4}{(x^2+y^2)^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}$$

$$D_yf(x,y) = \begin{cases} \dfrac{2x^4y}{(x^2+y^2)^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y) = (0,0) \end{cases}.$$

(0,0)

$x^2+y^2 \geq 2xy$

$\therefore \dfrac{2xy}{x^2+y^2} \leq 1$

Also, $\sqrt{x^2+y^2} < \delta$

Prove that f is differentiable at $(0,0)$.

We would like to check that $D_xf(x,y)$ and $D_yf(x,y)$ are continuous at $(0,0)$ to use Thm. 1.8.

- To check $D_xf(x,y)$ is cont. at $(0,0)$,

(1) $D_xf(0,0) = 0$

(2) Claim: $\lim\limits_{(x,y)\to(0,0)} D_xf(x,y) = 0$.

Let $\varepsilon > 0$. choose $\delta = \varepsilon > 0$ s.t. $|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2} < \delta$ and $|y| = \sqrt{y^2} \leq \sqrt{x^2+y^2} < \delta$, Then

$$\left|\frac{2xy^4}{(x^2+y^2)^2}\right| \leq \overbrace{\left|\frac{2xy}{x^2+y^2}\right|}^{1} \cdot \frac{|y^3|}{(x^2+y^2)} \leq \frac{(\sqrt{x^2+y^2})^2}{(x^2+y^2)} \cdot |y| = |y| < \delta = \varepsilon.$$

(3) $\therefore D_xf(0,0) = 0 = \lim\limits_{(x,y)\to(0,0)} D_xf(x,y)$.

To check $D_yf(x,y)$ is cont. at $(0,0)$

(1) $D_yf(x,y) = 0$

(2) Claim: $\lim\limits_{(x,y)\to(0,0)} D_yf(x,y) = 0$

Let $\varepsilon > 0$. choose $\delta = \sqrt[3]{\varepsilon} > 0$ s.t. $|x| < \sqrt{x^2} \leq \sqrt{x^2+y^2} < \delta$ and $|y| < \sqrt{y^2} \leq \sqrt{x^2+y^2} < \delta$, then

$$\left|\frac{2x^4y}{(x^2+y^2)^2}\right| \leq \overbrace{\left|\frac{2xy}{x^2+y^2}\right|}^{1} \cdot \frac{|x^3|}{(x^2+y^2)} \leq \frac{(\sqrt{x^2+y^2})^2}{x^2+y^2} \cdot |x| < \delta = \varepsilon$$

(3) $D_yf(0,0) = 0 = \lim\limits_{(x,y)\to(0,0)} D_yf(x,y)$.

$\therefore D_xf(x,y)$ and $D_yf(x,y)$ are continuous at $(0,0)$.

By Theorem 1.8, it implies that f is differentiable at $(0,0)$ #