Partial Derivative :

Def^n: Let f be a function of two variable ; $f=f(x,y)$.

The partial derivative of f with respect to x is defined by.

(x_0,y_0)

$$D_x f(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}.$$

and

$$D_y f(x,y) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x,y)}{\Delta y}.$$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \; f_x, f_y, \; u_x \; u_y, u_{xb}$

Ex: Let $f(x,y) = 3x^2 - 2xy + y^2$.

Sol: $$D_x f(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}$$

$$f = \begin{cases} \text{\_\_\_\_\_\_} & x = y \\ \text{\_\_\_\_\_\_} & x \neq y \end{cases}$$

$(x_0, y_0) \leftarrow (x_0+\Delta x, y_0)$ (x, y_0)

$(x_0, y_0) \in D_f$:

$$D_x f(x_0,y_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x}$$

$$D_y f(x_0,y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0+\Delta y) - f(x_0,y_0)}{\Delta y}$$

$x_0 + \Delta x = x$

$$D_x f(x_0,y_0) = \lim_{x \to x_0} \frac{f(x,y_0) - f(x_0,y_0)}{x - x_0}$$

$$D_y f(x_0,y_0) = \lim_{y \to y_0} \frac{f(x_0,y) - f(x_0,y_0)}{y - y_0}$$

Ex: $f(x,y) = 3x^2 - 2xy + y^2$; find $D_x f(3,-2)$

$$D_x f(3,-2) = \lim_{\Delta x \to 0} \frac{f(3+\Delta x, -2) - f(3,2)}{\Delta x}$$

$+12+4\Delta x$

$$= \lim_{\Delta x \to 0} \frac{3(3+\Delta x)^2 - 2(3+\Delta x)(-2) + (-2)^2}{\Delta x}$$

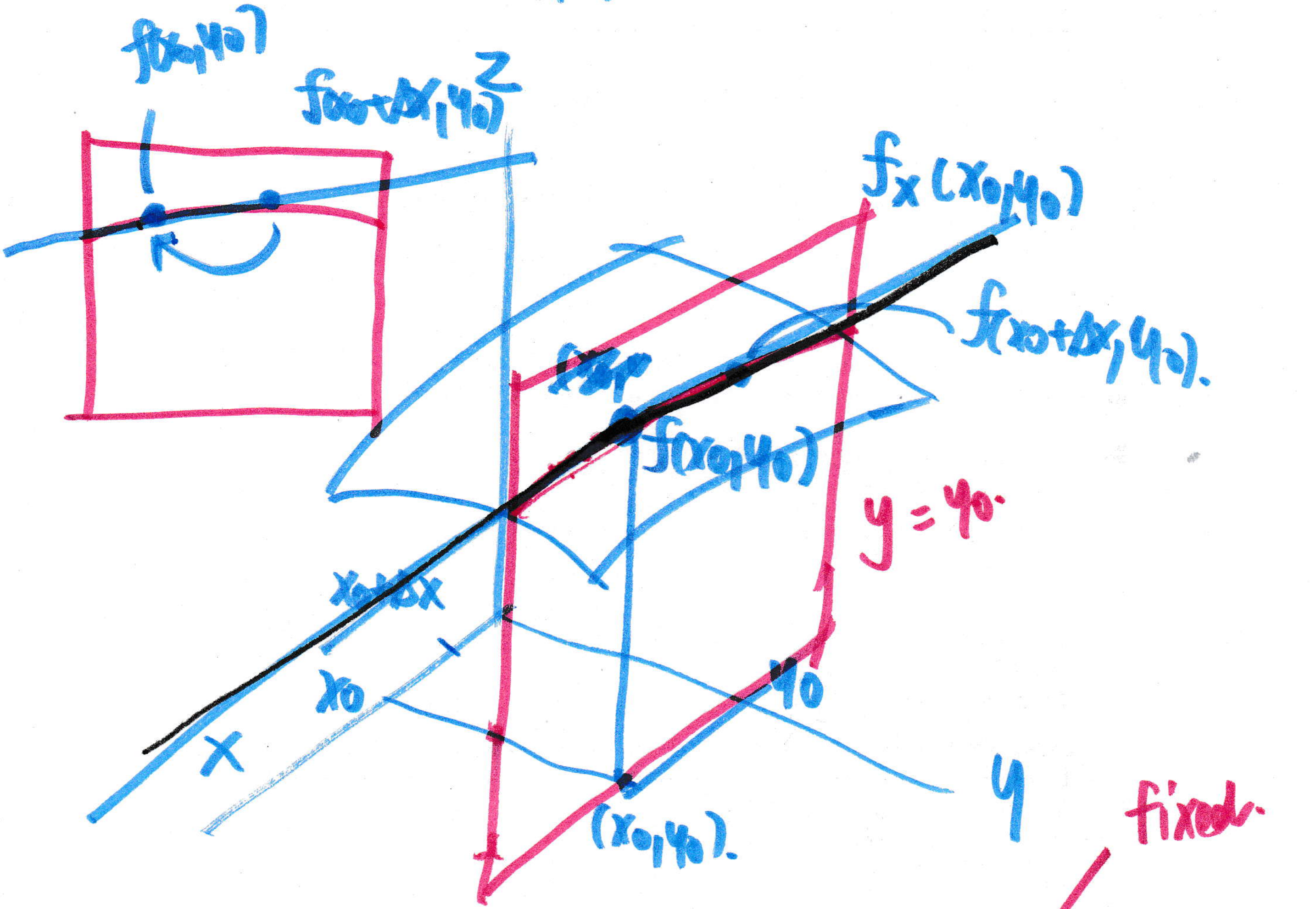
$$\frac{9 + 6\Delta x + \Delta x^2 - (3 \cdot 3^2 - 2 \cdot 3 \cdot (-2) + 4)}{\Delta x}$$

$f_x = 6x - 2y$

$f_x(3,-2) = 18 - (-4) = 22$

$$= \lim_{\Delta x \to 0} \frac{18\Delta x + 3\Delta x^2 + 4\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 18 + 3\Delta x + 4 = 22 \ \#$$



$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

Ex. 1.15 let $f(x,y) = 3x^2 - 4x^2y + 3xy^2 + 7x - 8y$

compute $D_x f(x,y)$ and $D_y f(x,y)$.

$D_x f(x,y) = 6x - 8xy + 3y^2 + 7$

$D_y f(x,y) = -4x^2 + 6xy - 8$.

Ex: 1.16 let $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$

Compute $f_x(0,y)$ and $f_y(x,0)$

$y \neq 0$; $f_x(0,y) = \lim_{\Delta x \to 0} \dfrac{f(0+\Delta x, y)^{*} - f(0,y)}{\Delta x}$

* Check!!!

$$= \lim_{\Delta x \to 0} \dfrac{\dfrac{\cancel{\Delta x} y(\Delta x^2 - y^2)}{\Delta x^2 + y^2} - 0}{\cancel{\Delta x}}$$

* If $y=0$.

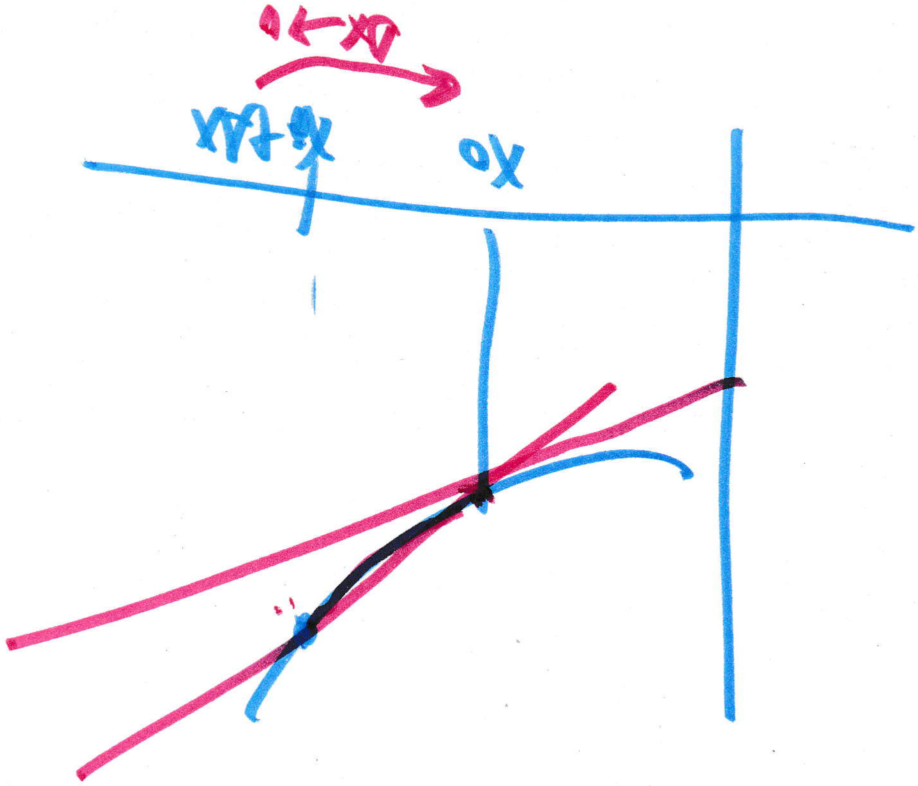
$$= \lim_{\Delta x \to 0} \dfrac{y(\Delta x^2 - y^2)}{\Delta x^2 + y^2} = -y \quad y \neq 0$$

$$= -y$$

$$y=0: \quad f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{0}{\Delta x} = 0$$

$$\therefore \; f_x(0,y) = \left.\begin{cases} -y \; ; & y \neq 0 \\ 0 \; ; & y = 0 \end{cases}\right\} = -y$$



$$f_y(x,0) = \lim_{\Delta y \to 0} \frac{f(x,\Delta y) - f(x,0)}{\Delta y}$$

$$= \lim_{\Delta y \to 0} \frac{\frac{x\Delta y(x^2-\Delta y^2)}{x^2+\Delta y^2} - 0}{\Delta y}$$

If $x \neq 0$; $f_y(x,0) = x$
If $x = 0$; $f_y(0,0) = 0$

$\left.\begin{array}{l} \\ \end{array}\right\} \therefore f_y(x,0) = x$

If $x \neq 0$, $f_y(x,0) = x$
$x = 0$ $f_y(0,0) = 1$

$$\rightarrow f_y(x,0) = \begin{cases} x, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

Ex. 1.17 Find the slope of the tangent line of the curve which is derived from the intersection of a surface $z = \frac{1}{2}\sqrt{24 - x^2 - 2y^2}$ and the plane $y = 2$ at the point $(2, 2, \sqrt{3})$

find $$\frac{\partial z}{\partial x} = \frac{1}{\cancel{2}} \cdot \frac{-\cancel{2}x}{2\sqrt{24 - x^2 - 2y^2}}$$

$$\therefore \left.\frac{\partial z}{\partial x}\right|_{(2,2,\sqrt{3})} = \frac{-\cancel{2}}{\cancel{2}\sqrt{24 - 2^2 - 2 \cdot 2^2}} = \frac{-1}{2\sqrt{3}}$$

Remark: If a function f has continuous partial derivative $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ in a region, then f must be continuous in the region.

However, the existence of the partial derivative alone is not guarantee the continuity of f.

Ex: $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ; \quad (x,y) \neq (0,0) \\ 0 & ; \quad (x,y) = (0,0) \end{cases}$

Prove that (1) $f_x(0,0)$, $f_y(0,0)$ exist.

(2) $f(x,y)$ is discontinuous at $(0,0)$

① $f_x(0,0) = \lim\limits_{\Delta x \to 0} \dfrac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0$

$f_y(0,0) = \lim\limits_{\Delta y \to 0} \dfrac{f(0, \Delta y) - f(0,0)}{\Delta y} = 0$

②. $f(0,0) = 0$. ; Consider $\lim\limits_{(x,y) \to (0,0)} f(x,y)$

$\lim\limits_{x \to 0} \lim\limits_{y \to 0} \dfrac{xy}{x^2+y^2} = 0 = \lim\limits_{y \to 0} \lim\limits_{x \to 0} \dfrac{xy}{x^2+y^2}$

③ $f(0,0) = 0$. Consider $\lim_{(x,y)\to(0,0)} f(x,y)$

Check: $y = x$

$$\lim_{\substack{(x,y)\to(0,0) \\ y=x}} f(x,y) = \lim_{x\to 0} f(x,x)$$

$$= \lim_{x\to 0} \frac{\cancel{x^2}}{2\cancel{x^2}} = \frac{1}{2}$$

$y = mx$