f cont at $P = P_0 \Leftrightarrow \forall \varepsilon > 0 \ \exists \delta > 0$

$$\boxed{0 \leq \| P - P_0 \| < \delta \Rightarrow | f(P) - f(P_0) | < \varepsilon}$$

Generalization to higher dimension

Let $P(x_1, \ldots, x_n) \in \mathbb{R}^n$; $f = f(x_1, \ldots, x_n)$.

$$D_{x_k} f(x_1, x_2, \ldots, x_n) = \lim_{\Delta x_k \to 0} \frac{f(x_1, \ldots, x_{k-1}, x_k + \Delta x_k, x_{k+1}, \ldots, x_n) - f(x_1, \ldots x_n)}{\Delta x_k}$$

Ex Let $f(x, y, z) = x^2 y + y z^2 + z^3$

Verify that $x f_x(x,y,z) + y f_y(x,y,z) + z f_z(x,y,z) = 3 f(x,y,z)$

Proof $x f_x(x,y,z) + y f_y(x,y,z) + z f_z(x,y,z)$

$$= x(2xy) + y(x^2 + z^2) + z(2yz + 3z^2)$$

$$= 2x^2y + x^2y + yz^2 + 2yz^2 + 3z^3$$

$$= 3x^2y + 3yz^2 + 3z^2$$

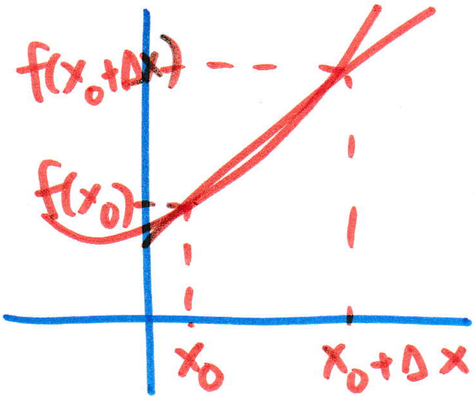
$$= 3(x^2y + yz^2 + z^3)$$

$$= 3f(x,y,z)$$

#

1.5 Total Differential and Differentiability.

in $\boxed{1D}$ $f(x)$ is differentiable at $x_0 \Longleftrightarrow$



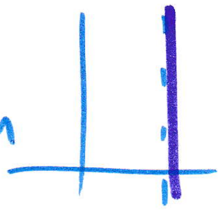
$$f'(x_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x) - f(x_0)}{\Delta x} \quad -(*)$$

มีกรณีไหนที่ $f'(x)$ ไม่มีอ่ะ

~~If~~ limit $(*)$ exists, then f is differentiable at x_0.

Consequence Results

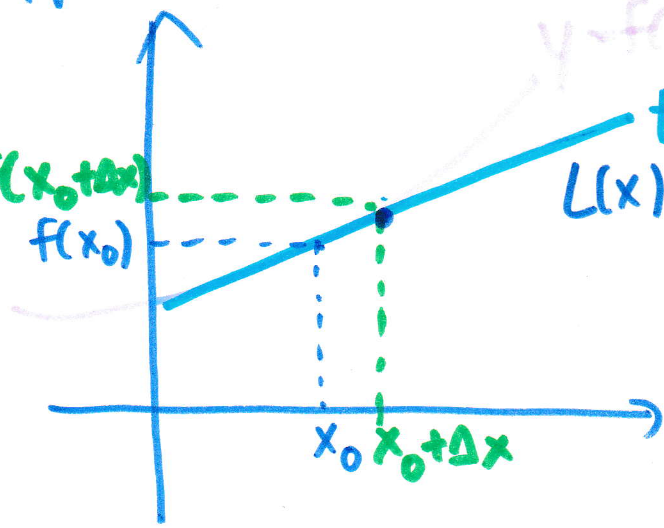
① Thm If f is differentiable at x_0, then f is continuous at x_0.



② f has nonvertical tangent line at $(x_0, f(x_0))$

③ Local approximation

f can be approximated by the local linear approximation at x_0.



ต้องการหา $f(x_0+\Delta x)$

tangent line at x_0

$$L(x) = f(x_0) + f'(x_0)(x - x_0)$$

Question : Can we extend these consequences in more general dimension?

If f is differentiable at (x_0, y_0), then

(1) f is continuous at (x_0, y_0) ?

(2) f has nonvertical tangent **plane** at $(x_0, y_0, f(x_0, y_0))$

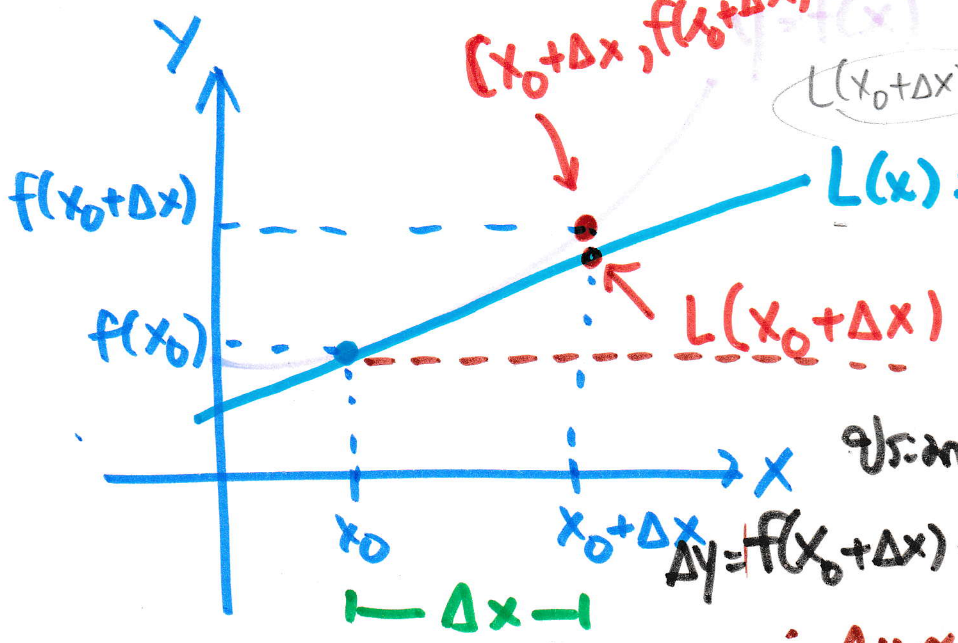
(3) f can be approximated by the local ? linear function at $(x_0, y_0, f(x_0, y_0))$?

$\boxed{Q:}$ ถ้าอนุพันธ์ย่อยทั้งสองค่ามีค่า ของ f ที่ (x_0, y_0) แล้ว f จะต่อเนื่องที่ (x_0, y_0)

$\boxed{A:}$ ไม่จริง! ตัวอย่างค้าน

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$$

Increments ใน 1D



$L(x_0+\Delta x) = f(x_0) + f'(x_0) + (x_0+\Delta x - x_0)$

$L(x) = f(x_0) + f'(x_0)(x - x_0)$

$\Delta y = f(x_0+\Delta x) - f(x_0)$ ← increments.

ประมาณ

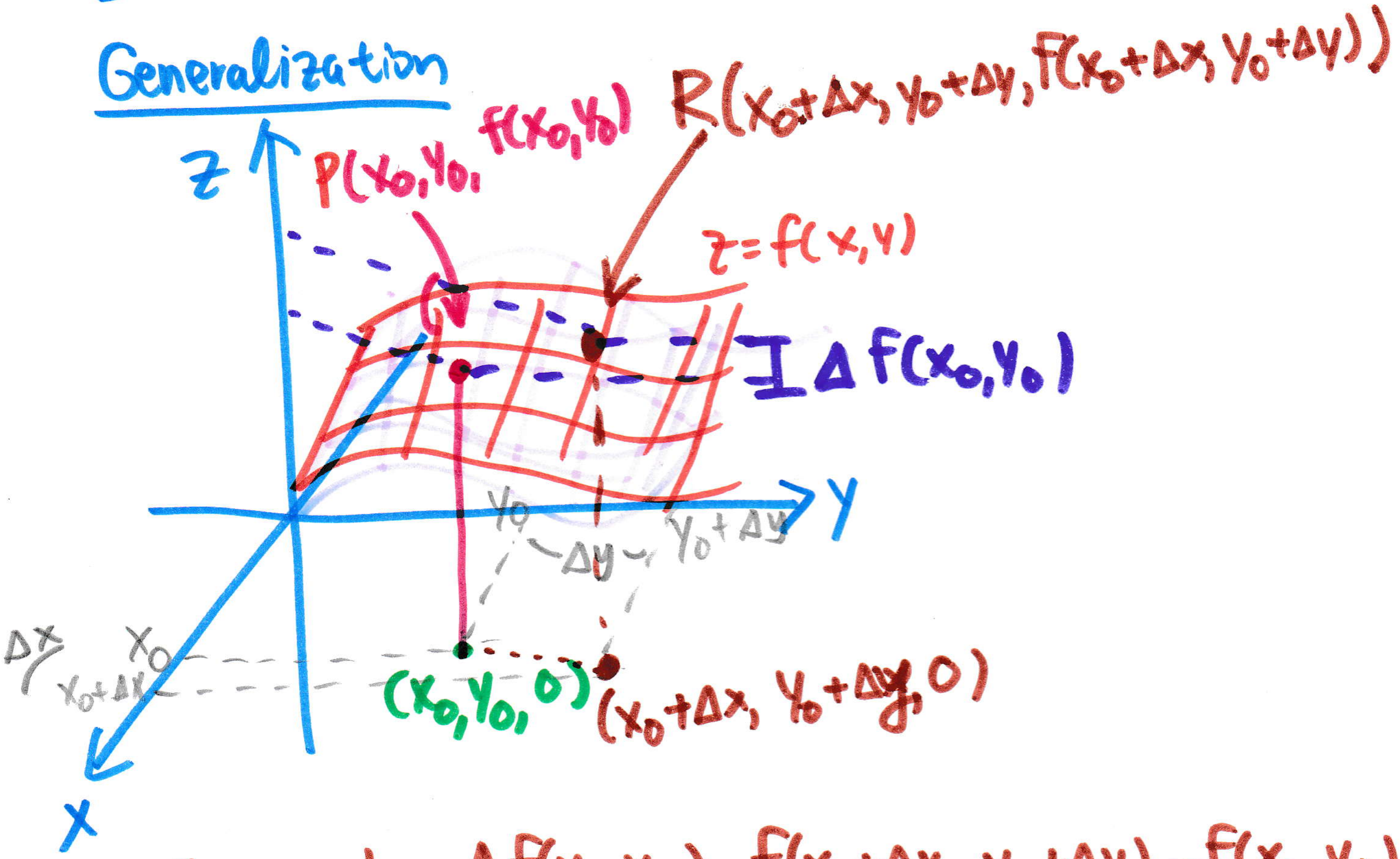
$$\Delta y = f(x_0+\Delta x) - f(x_0) \approx L(x_0+\Delta x) - f(x_0)$$
$$= f'(x_0)\Delta x$$

$\therefore \Delta y \approx f'(x_0)\Delta x_0$

∴ จริงๆแล้ว $\Delta y = f'(x_0)\Delta x_0 + \underbrace{err}_{\eta(\Delta x)}$

เมื่อ $\Delta x \to 0$ ได้ว่า $\eta \to 0$

Generalization



Increments $\Delta f(x_0,y_0) = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$

Ex 1.19 $f(x,y) = 3x - xy^2$. Find increments of f at (x_0,y_0).

Sol^n

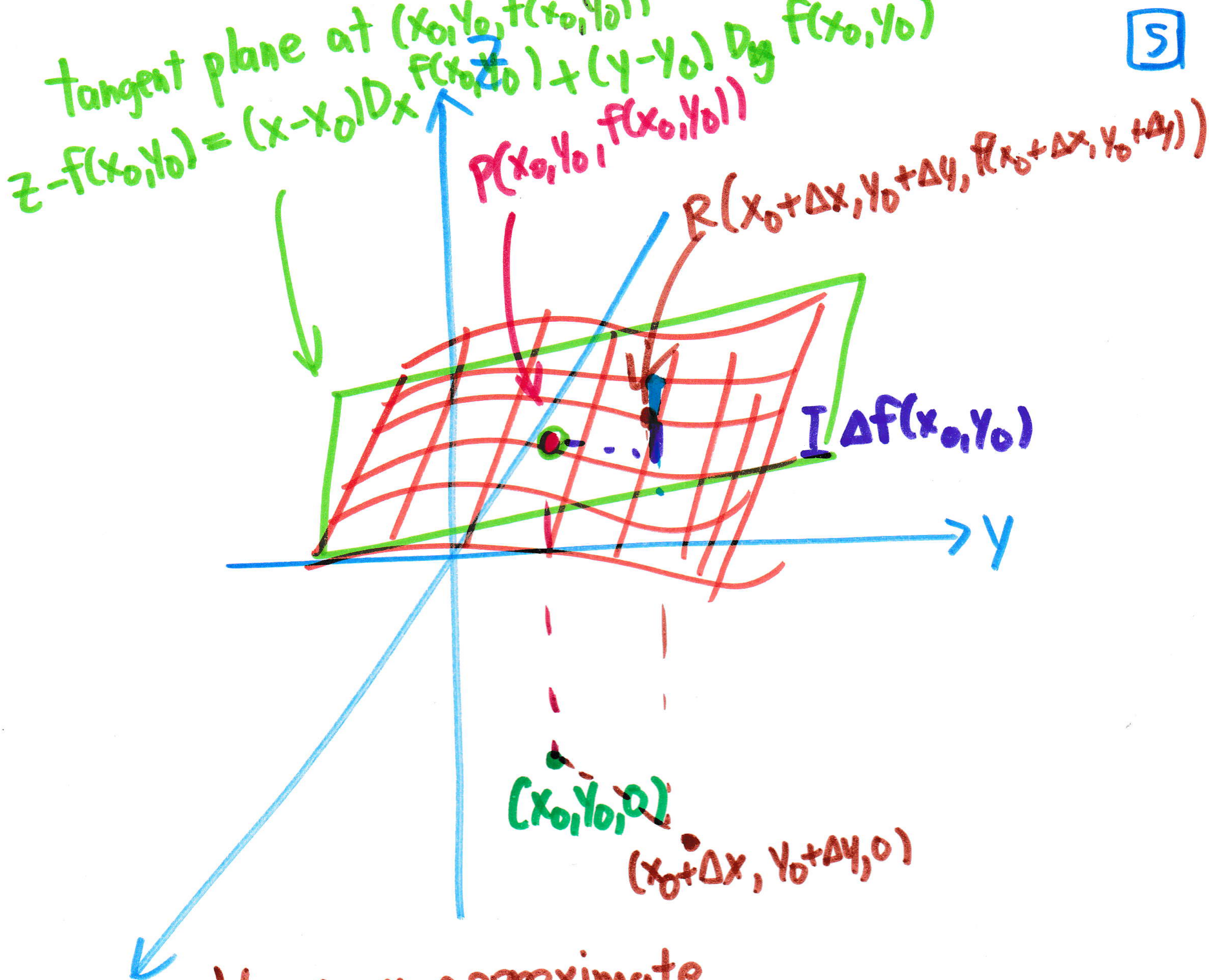
$$\Delta f(x_0,y_0) = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$$

$$= \left[3(x_0+\Delta x) - (x_0+\Delta x)(y_0+\Delta y)^2\right] - (3x_0 - x_0y_0^2)$$

$$= \left[\cancel{3x_0} + 3\Delta x - (x_0+\Delta x)(y_0^2 + 2y_0\Delta y + (\Delta y)^2)\right] - \cancel{3x_0} + x_0y_0^2$$

$$= 3\Delta x - \{x_0y_0^2 - 2x_0y_0\Delta y - x_0(\Delta y)^2 - \Delta x y_0^2 - 2y_0\Delta x\Delta y$$
$$- \Delta x(\Delta y)^2 + x_0y_0^2$$

$$= 3\Delta x - y_0^2\Delta x - 2x_0y_0\Delta y - 2y_0\Delta x\Delta y - x_0(\Delta y)^2$$
$$- \Delta x(\Delta y)^2 \quad \#$$



We may approximate

$$\Delta f(x_0, y_0) \approx D_x f(x_0, y_0)\Delta x + D_y f(x_0, y_0)\Delta y.$$

$\Downarrow$ with some error $\varepsilon_1, \varepsilon_2$ ($\varepsilon_1, \varepsilon_2$ are function of $\Delta x, \Delta y$)

$$\boxed{\Delta f(x_0, y_0) = D_x f(x_0, y_0)\Delta x + D_y f(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y}$$

Definition Let $z=f(x,y)$ be a function, and $D_x f(x_0,y_0)$ and $D_y f(x_0,y_0)$ exist.
f has total differential or differentiable at (x_0,y_0), if at (x_0,y_0)

$$\Delta f(x_0,y_0) = \boxed{D_x f(x_0,y_0)\Delta x} + \boxed{D_y f(x_0,y_0)\Delta y} + \varepsilon_1 \Delta x + \boxed{\varepsilon_2 \Delta y}$$

where $\varepsilon_1, \varepsilon_2$ are function of $\Delta x, \Delta y$ s.t.

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_1 = 0 \quad \text{and} \quad \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_2 = 0.$$

Ex Let $f(x,y)=3x-xy^2$. Show that f is differentiable everywhere in $\mathbb{R}^2$.

$$D_x f(x,y) = 3-y^2, \quad D_y f(x,y) = -2xy.$$

$$\Delta f(x_0,y_0) = \boxed{3\Delta x - y_0^2\Delta x} \boxed{-2x_0y_0\Delta y} - 2y_0\Delta x\Delta y - x_0(\Delta y)^2 - \Delta x(\Delta y)^2$$

$$= \underbrace{(3-y_0^2)}_{D_x f(x_0,y_0)}\Delta x + \underbrace{(-2x_0y_0)}_{D_y f(x_0,y_0)}\Delta y + (-2y_0\Delta y)\Delta x + \boxed{(-x_0\Delta y - \Delta x\Delta y)\Delta y}$$

เลือก $\varepsilon_1 = -2y_0\Delta y \Rightarrow \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_1 = 0$

$\varepsilon_2 = -x_0\Delta y - \Delta x\Delta y \Rightarrow \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_2 = 0$

หรือเลือก $\Delta f(x_0,y_0) = (3-y_0^2)\Delta x + (-2x_0y_0)\Delta y + (\underbrace{-2y_0\Delta y-(\Delta y)^2}_{\varepsilon_1})\Delta x + (\underbrace{x_0\Delta y}_{\varepsilon_2})\Delta y$

หรือเลือก $\Delta f(x_0,y_0) = (3-y_0^2)\Delta x + (-2x_0y_0)\Delta y + (\underbrace{-(\Delta y)^2}_{\varepsilon_1})\Delta x + (\underbrace{-2y_0\Delta x - x_0\Delta y}_{\varepsilon_2})\Delta y$

ทุกคู่ที่เลือก $\lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_1 = 0 = \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_2$

[Anton] A function f of two variables is s.t.b. differentiable at (x_0, y_0) provided that $D_x f(x_0,y_0)$ and $D_y f(x_0,y_0)$ exist, and

$$\lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta f(x_0,y_0) - D_x f(x_0,y_0)\Delta x - D_y f(x_0,y_0)}{\sqrt{(\Delta x)^2+(\Delta y)^2}} = 0.$$

เตือน

$$\boxed{f \text{ is differentiable at } x = x_0} \;\to\; \boxed{f \text{ is continuous at } x = x_0}$$

$$\boxed{f \text{ is differentiable at } x = x_0} \;\not\leftarrow\; \boxed{f \text{ is continuous at } x = x_0}$$

$f(x) = |x|$

ธรรมชาติ ของ differentiable at (x_0, y_0)

① continuous $(x,y)=(x_0,y_0)$

② partial derivative

$$\boxed{f \text{ is differentiable at } (x,y)=(x_0,y_0)} \;\xrightarrow{?}\; \boxed{f \text{ is continuous at } (x,y)=(x_0,y_0)}$$

$$\boxed{f \text{ is differentiable at } (x,y)=(x_0,y_0)} \;\xleftarrow{?}\; \boxed{f \text{ is continuous at } (x,y)=(x_0,y_0)}$$

$$\boxed{f \text{ is differentiable at } (x_0,y_0)} \;\xrightarrow{?}\; \boxed{f \text{ has partial derivative at } (x_0,y_0)}$$

$$\boxed{f \text{ is differentiable at } (x_0,y_0)} \;\xleftarrow{?}\; \boxed{f \text{ has partial derivative at } (x_0,y_0)}$$

$$\boxed{f \text{ has partial derivative at } (x_0,y_0)} \;\xrightarrow{?}\; \boxed{f \text{ is continuous at } (x,y)=(x_0,y_0)}$$

$$\boxed{f \text{ has partial derivative at } (x_0,y_0)} \;\xleftarrow{?}\; \boxed{f \text{ is continuous at } (x,y)=(x_0,y_0)}$$

การบ้าน อ่าน Theorem 1.7.