$$\Delta f(x_0, y_0) = A\Delta x + B\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Thm f is differentiable at (x_0, y_0)

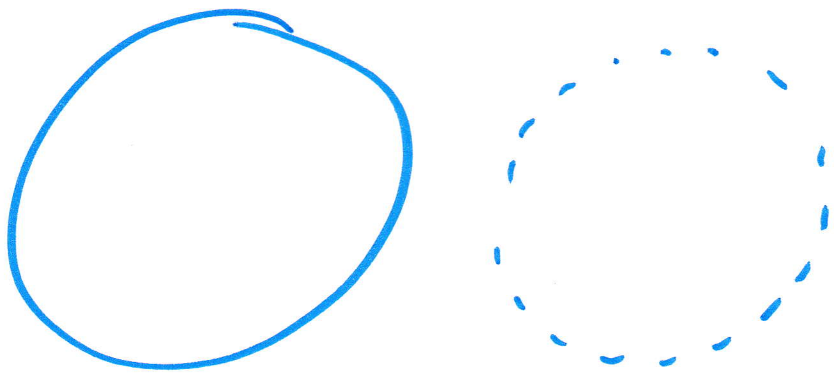
$$\Downarrow$$

f has partial derivative at (x_0, y_0)

and $A = D_x f(x_0, y_0)$

$B = D_y f(x_0, y_0)$

$\mathbb{R}^2$



f differentiable at (x_0,y_0)

$$\Delta f(x_0,y_0) = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) \text{ — incremental}$$

$$\left[\ \Delta f(x_0,y_0) = D_x f(x_0,y_0)\,\Delta x + D_y f(x_0,y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y \right.$$

$$\left. \text{เมื่อ } \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_1 = 0,\ \lim_{(\Delta x,\Delta y)\to(0,0)} \varepsilon_2 = 0\ \right]$$

$\boxed{f \in C^1}$ (มีอนุพันธ์ย่อยที่ต่อเนื่อง)

$\boxed{f \in C^1} \Rightarrow \boxed{\text{f is differentiable at } (x_0,y_0)}$

$\boxed{\text{f is differentiable at } (x_0,y_0)} \not\Rightarrow \boxed{f \in C^1}$

$\boxed{\text{f is differentiable at } (x_0,y_0)} \xrightarrow{\text{Thm 1.6}} \boxed{\text{f is continuous at } (x_0,y_0)}$

$\boxed{\text{f is continuous at } (x_0,y_0)} \not\Rightarrow \boxed{\text{f is differentiable at } (x_0,y_0)}$: $f(x,y) = |x|(1+y)$ at $(0,0)$

$\boxed{\text{f is differentiable at } (x_0,y_0)} \xrightarrow{\text{Def}^{\text{n}}} \boxed{\text{f has partial derivative at } (x_0,y_0)}$

$\boxed{\text{f has partial derivative at } (x_0,y_0)} \not\Rightarrow \boxed{\text{f is differentiable at } (x_0,y_0)}$: $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ;(x,y)\neq(0,0) \\ 0 & ;(x,y)=(0,0) \end{cases}$ at $(0,0)$

$\boxed{\text{f has partial derivative at } (x_0,y_0)} \not\Rightarrow \boxed{\text{f is continuous at } (x_0,y_0)}$

$\boxed{\text{f is continuous at } (x_0,y_0)} \not\Rightarrow \boxed{\text{f has partial derivative at } (x_0,y_0)}$: $f(x,y) = |x|(1+y)$ at $(0,0)$

Thm 1.6 If $f = f(x,y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0)

Proof [**CLAIM** f is continuous at (x_0, y_0)

$$\Updownarrow$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

$$\Updownarrow$$

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) - f(x_0,y_0) = 0 \quad]$$

Let f be differentiable at (x_0, y_0).

Hence, at $(x_0,y_0) \in \mathbb{R}^2$, we can find $\varepsilon_1, \varepsilon_2$ s.t.

$$\boxed{\Delta f(x_0,y_0)} = D_x f(x_0,y_0)\Delta x + D_y f(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

where $\varepsilon_1, \varepsilon_2 \to 0$ when $(\Delta x, \Delta y) \to (0,0)$

Since $\Delta f(x_0,y_0) = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$, we have

$$f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0) = D_x f(x_0,y_0)\Delta x + D_y f(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

If we take lim when $\Delta x, \Delta y \to 0$

$$\therefore \lim_{(\Delta x,\Delta y)\to(0,0)} [f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)] = 0$$

Since $\Delta x = x - x_0$, $\Delta y = y - y_0$, then $x \to x_0$ and $y \to y_0$ if $\Delta x \to 0$ and $\Delta y \to 0$

Thus. $\lim_{(x,y)\to(x_0,y_0)} f(x,y) - f(x_0,y_0) = 0$, That is,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$$

$\therefore f$ is continuous at (x_0,y_0) #

Ex $f(x,y) = |x|(1+y)$ is continuous but not differentiable. at $(0,0)$

- $f(x,y)$ is continuous (Ex 1.12)
- f is not differentiable. ?!?

Check จากอนุพันธ์ย่อย

$$D_x f(0,0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{|\Delta x|(1+0) - 0}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x}$$

$$f(x) = \frac{|x|}{x} = \begin{cases} 1 ; & x > 0 \\ -1 ; & x < 0 \end{cases}$$

พิจารณา $\displaystyle\lim_{\Delta x \to 0^+} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^+} 1 = 1$

$\displaystyle\lim_{\Delta x \to 0^-} \frac{|\Delta x|}{\Delta x} = \lim_{\Delta x \to 0^-} -1 = -1$

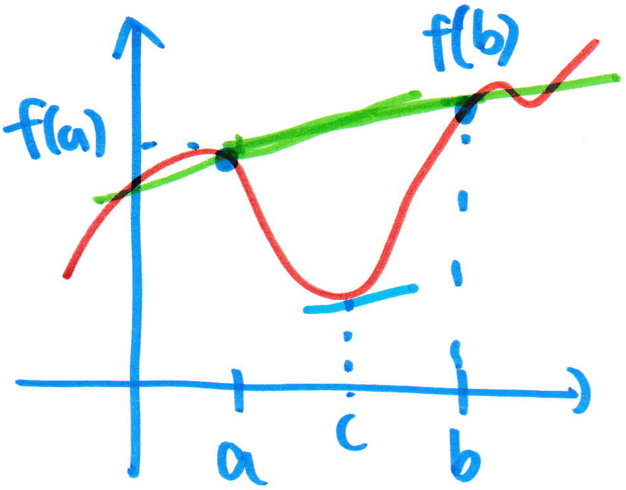
เนื่องจาก $\displaystyle\lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x}$ DNE,

$D_x f(0,0)$ DNE.

ดังนั้น f ~~is not~~ หาอนุพันธ์ไม่ได้ที่ $(0,0)$. #

Thm MVT (Single variable)

Let f be a continuous on $[a,b]$ and differentiable on (a,b). Then $\exists\, c \in (a,b)$



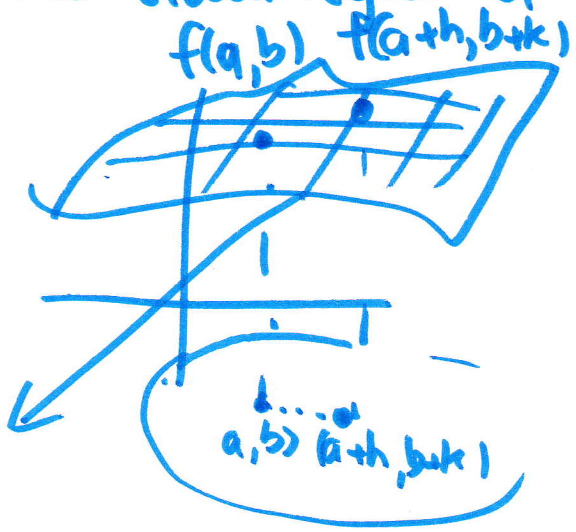
$$\frac{f(b)-f(a)}{b-a} = f'(c)$$

Thm 1.7 Mean Value Theorem of a function of two variables

If $f(x,y)$ is continuous in a closed region of $\mathbb{R}^2$ and $D_x f(x,y)$, $D_y f(x,y)$ exist in their open region, then

$$f(a+h, b+k) - f(a,b) = h D_x f(a+\theta h, b+\theta k) + k D_y f(a+\theta h, b+\theta k)$$

for some $\theta \in (0,1)$ and $(a+h, b+k)$ is a point in the closed region of $\mathbb{R}^2$.



Ex $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$

① $D_x f(0,0)$, $D_y f(0,0)$ exist. (ทำแล้วจ้า) by A.J. Ben.

② CHECK ว่า $f(0,0)$ cont ? (ทำแล้วจ้า) by A.J. Ben.

Theorem 1.8 Let $f = f(x,y)$. Suppose $D_x f$ and $D_y f$ is in $N(P_0; r)$ s.t. $P_0 = (x_0, y_0)$

If $D_x f$ and $D_y f$ are continuous at P_0, then f is ~~differentiable~~ differentiable at P_0.

Proof CLAIM f is differentiable at P_0

$$\Updownarrow$$

$$\Delta f(x_0,y_0) = D_x f(x_0,y_0)\Delta x + D_y f(x_0,y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

เมื่อ $\varepsilon_1, \varepsilon_2 \to 0$ เมื่อ $(\Delta x, \Delta y) \to (0,0)$

Choose a point $(x_0+\Delta x, y_0+\Delta y) \in N(P_0; r)$

$$\Delta f(x_0,y_0) = f(x_0+\Delta x, y_0+\Delta y) - f(x_0,y_0)$$
$$= [f(x_0+\Delta x, y_0+\Delta y) - f(x_0+\Delta x, y_0)] + [f(x_0+\Delta x, y_0) - f(x_0,y_0)] \quad \text{— } \star$$

Since $D_x f, D_y f \in N(P_0; r)$, $(x_0+\Delta x, y_0+\Delta y) \in N(P_0; r)$ by MVT.

$\exists\, \xi_1 \in (y_0, y_0+\Delta y)$ s.t.

Fix x $\quad f(x_0+\Delta x, y_0+\Delta y) - f(x_0+\Delta x, y_0) = D_y f(x_0+\Delta x, \xi_1)\Delta y.$

$\xi_1 = y_0 + \theta_1 \Delta y$
~~for some $\theta_1 \in (0,1)$~~

$\exists\, \xi_2 \in (x_0, x_0+\Delta x)$ s.t.

Fix y $\quad f(x_0+\Delta x, y_0) - f(x_0, y_0) = D_x f(\xi_2, y_0)\Delta x$

$\xi_2 = x_0 + \theta_2 \Delta x$ for some $\theta_2 \in (0,1)$

$\therefore$ From ⭐, we have

$$\boxed{\Delta f(x_0,y_0) = D_y f(x_0+\Delta x, \xi_1)\Delta y + D_x f(\xi_2, y_0)\Delta x.}$$

Since $(x_0+\Delta x, y_0+\Delta y) \in N(P_0; r)$ and $D_x f, D_y f$ cont. at (x_0, y_0)

Then $\displaystyle\lim_{(\Delta x, \Delta y)\to(0,0)} D_y f(x_0+\Delta x, \xi_1) = D_y f(x_0, y_0)$

and $\displaystyle\lim_{(\Delta x, \Delta y)\to(0,0)} D_x f(\xi_2, y_0) = D_x f(x_0, y_0)$

Choose $\varepsilon_1 = D_y f(x_0+\Delta x, \xi_1) - D_y f(x_0, y_0)$

Then $\displaystyle\lim_{(\Delta x, \Delta y)\to(0,0)} \varepsilon_1 = 0$

$\varepsilon_2 = D_x f(\xi_2, y_0) - D_x f(x_0, y_0)$

Then $\displaystyle\lim_{(\Delta x, \Delta y)\to(0,0)} \varepsilon_2 = 0$

$$\therefore \Delta f(x_0,y_0) = [D_y f(x_0,y_0) + \varepsilon_1]\Delta y + [D_x f(x_0,y_0) + \varepsilon_2]\Delta x$$

$$= D_y f(x_0,y_0)\Delta y + D_x f(x_0,y_0)\Delta x + \varepsilon_1 \Delta y + \varepsilon_2 \Delta x$$

when $\varepsilon_1, \varepsilon_2 \to 0$ where $(\Delta x, \Delta y) \to (0,0)$. $\therefore f$ is differentiable at (x_0, y_0) #

Ex Let $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = 0 \end{cases}$ Ex

Prove that f is differentiable at $(0,0)$

① หาอนุพันธ์ย่อย

$$D_x f(0,0) = \lim_{\Delta x \to 0} \frac{f(0+\Delta x, 0) - f(0,0)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\frac{(\Delta x)^2(0)}{x^2+y^2} - 0}{\Delta x} = 0$$

Similary, $D_y f(0,0) = 0$

ที่ (x,y) โดยที่ $(x,y) \neq (0,0)$

$$D_x f(x,y) = \frac{(x^2+y^2)(2xy^2) - (x^2y^2)(2x)}{(x^2+y^2)^2}$$

$$= \frac{\cancel{2x^3y^2} + 2xy^4 - \cancel{2x^3y^2}}{(x^2+y^2)^2}$$

$$= \frac{2xy^4}{(x^2+y^2)^2}$$

$$D_y f(x,y) = \frac{2x^4y}{(x^2+y^2)^2}$$

$$D_x f(x,y) = \begin{cases} \dfrac{2xy^4}{(x^2+y^2)^2} & ; (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}, \quad D_y f(x,y) = \begin{cases} \dfrac{2x^4y}{(x^2+y^2)^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

② <u>CHECK</u> $D_x f(x,y)$, $D_y f(x,y)$ is cont. at $(0,0)$

HW

<u>Ex</u> Let
$$f(x,y)=\begin{cases}(x^2+y^2)\cdot \sin \dfrac{1}{\sqrt{x^2+y^2}} & ; (x,y)\neq(0,0)\\ 0 & ; (x,y)=(0,0)\end{cases}$$

Prove that f is differentiable, but not continuously differentiable at $(0,0)$.

Hint
① หา $D_x f(0,0)$, $D_y f(0,0)$
② Check $\underbrace{D_x f(x,y),\ D_y f(x,y)}_{\text{ไม่ต่อเนื่อง}}$ cont ?
③ ใช้นิยามของ differentiable.