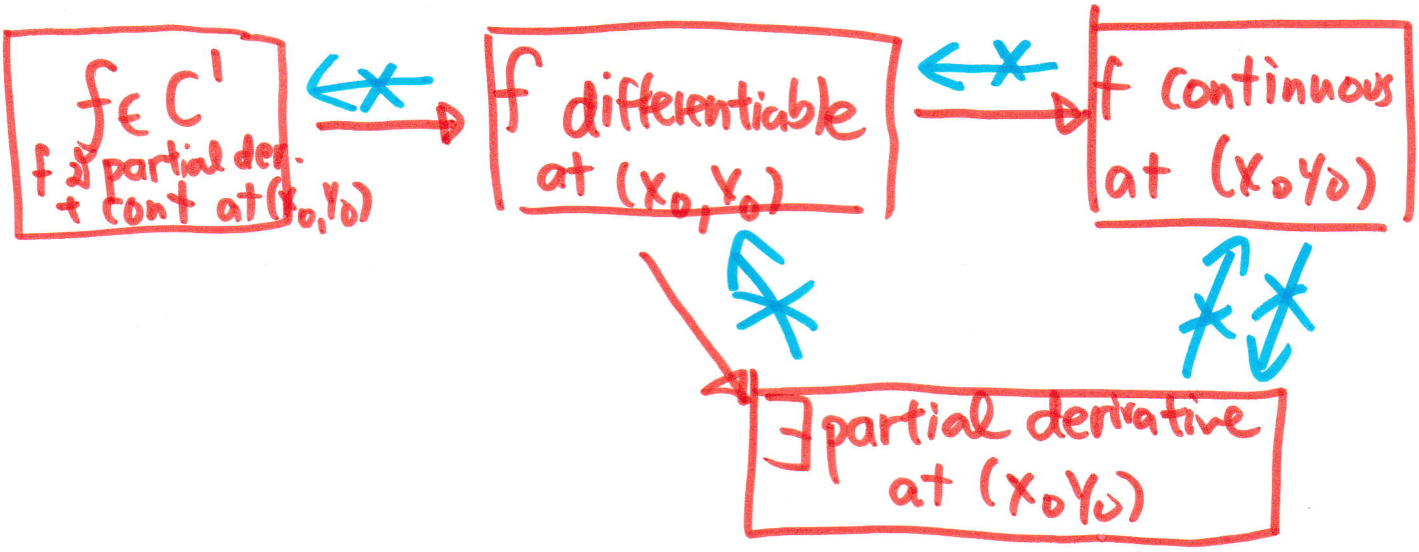
Ex $f(x,y)=\begin{cases}\dfrac{x^2y^2}{x^2+y^2} & ; (x,y)\neq(0,0)\\ 0 & ; (x,y)=(0,0)\end{cases}$

is differentiable at $(0,0)$

① หา partial derivative $[D_x f(x,y), D_y f(x,y)]$

② ตรวจสอบว่า $D_x f(0,0), D_y f(0,0)$ cont ?

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$$\Delta f(x_0,y_0) = \underbrace{D_x f(x_0,y_0)\,\Delta x + D_y f(x_0,y_0)\,\Delta y}_{\text{principle part}} + \underline{\varepsilon_1 \Delta x + \varepsilon_2 \Delta y}$$

s.t. $\varepsilon_1, \varepsilon_2 \to 0$ when $(\Delta x, \Delta y) \to (0,0)$.

$$\boxed{dz = df(x,y,\Delta x,\Delta y) = D_x f(x,y)\,\Delta x + D_y f(x,y)\,\Delta y} \quad \text{— } (*)$$

Suppose that $z = f(x,y) = x$ ; $D_x f(x,y) = 1$, $D_y f(x,y) = 0$

$(*)$ $dx = D_x f(x,y)\,\Delta x + D_y f(x,y)\,\Delta y$

$\therefore dx = \Delta x$

Similarly, $z = f(x,y) = y$ ; $D_x f(x,y) = 0$, $D_y f(x,y) = 1$

$dy = D_x f(x,y)\,\Delta x + D_y f(x,y)\,\Delta y$

$\therefore dy = \Delta y$

$$\therefore \boxed{dz = D_x f(x,y)\,dx + D_y f(x,y)\,dy}$$

เมื่อเราประมาณ $\Delta f(x_0,y_0) \approx df$

total differential

ดิฟเฟอเรนเชียลรวมยอด

-(ผลต่างเชิงอนุพันธ์รวม)

$$\boxed{dz = D_x f(x_0,y_0)\,dx + D_y f(x_0,y_0)\,dy}$$

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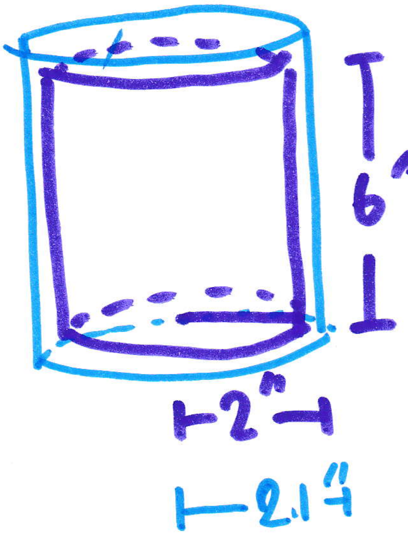
**Ex** กระป๋องใบหนึ่งมีฝาปิด โดยที่มีความสูงภายใน 6 นิ้ว, รัศมีภายใน 2 นิ้ว หนา 0.1 นิ้ว ในการผลิตนี้จะคิดเงิน ลบ.นิ้วละ 10 บาท จงประมาณว่าต้องจ่ายเท่าไร.




$\Delta V = V_{นอก} - V_{ใน}$

$|dr| = 0.1$

$|dh| = 0.2$

อยากได้ $\Delta V$

$\Delta V$ จะประมาณได้ด้วย $dV$.

$$dV = \frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial h}dh.$$

$f(r,h) = V = \pi r^2 h$

$\frac{\partial f}{\partial r} = 2\pi r h$

$\frac{\partial f}{\partial h} = \pi r^2$

มอง $\left.\frac{\partial f}{\partial r}\right|_{(r,h)=(2,6)}$, $\left.\frac{\partial f}{\partial h}\right|_{(r,h)=(2,6)}$

$$|dV| = \left|\frac{\partial f}{\partial r}dr + \frac{\partial f}{\partial h}dh\right|$$

$$\le \left|\frac{\partial f}{\partial r}\right|\cdot|dr| + \left|\frac{\partial f}{\partial h}\right|\cdot|dh|$$

$$\le |2\pi r h|\cdot|dr| + |\pi r^2|\cdot|dh|$$

$$= (2\pi)(2)(6)(0.1) + (\pi)(4)(0.2)$$

$$= 3.2\pi \text{ (นิ้ว)}^3$$

$\therefore$ ราคาโดยประมาณเท่ากับ $32\pi$ บาท #

## Generalized Dimension: $P=(x_1,\dots,x_n)$

$f=f(x_1,\dots,x_n)$

$$\boxed{\Delta f(P)=f(x_1+\Delta x_1, x_2+\Delta x_2,\dots,x_n+\Delta x_n)-f(x_1,\dots,x_n)}$$

## Differentiability

$$\Delta f(P)=D_{x_1}f(P)\Delta x_1+D_{x_2}f(P)\Delta x_2+\dots+D_{x_n}f(P)\Delta x_n+$$
$$\varepsilon_1\Delta x_1+\varepsilon_2\Delta x_2+\dots+\varepsilon_n\Delta x_n$$

เมื่อ $\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n\to 0$ เมื่อ $(\Delta x_1,\Delta x_2,\dots,\Delta x_n)\to(0,0,\dots,0)$

① $f$ differentiable at $P_0 \Longrightarrow P$ continuous at $P_0$

② $f$ differentiable if $\exists D_{x_1}f, D_{x_2}f,\dots,D_{x_n}f\in N(P,r)$ s.t. $D_{x_1}f,\dots,D_{x_n}f$ cont.

③ $D_{x_1}f,\dots D_{x_n}f,\dots,D_{x_n}f \not\Longrightarrow f$ differentiable.

## Total Differential

$$df(P,\Delta x_1,\dots,\Delta x_n)=D_{x_n}f(P)\Delta x_1+\dots+D_{x_n}f(P)\Delta x_n$$

$$\boxed{dW=\frac{\partial W}{\partial x_1}dx_1+\dots+\frac{\partial W}{\partial x_n}dx_n.}$$

# 1.6 Euler's Theorem for Homogeneous Function

**Def$^n$** Let $F = f(x_1, \dots, x_n)$ be a function $f$ is s.t.b. (said to be) a **homogeneous function** of degree $m$, For all values of the parameter $\lambda > 0$ and some constant $m$ if

$$f(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^m f(x_1, \dots x_n).$$

**Ex** Let $f(x,y) = x^4 - 2xy^3 - 5y^4$

Is $f$ a homogeneous function?

$\boxed{\text{CLAIM}}$ $\quad f(\lambda x, \lambda y) = \lambda^m f(x,y)$

**Sol$^n$** Let $\lambda > 0$.

$$\begin{aligned} f(\lambda x, \lambda y) &= (\lambda x)^4 - 2(\lambda x)(\lambda y)^3 - 5(\lambda y)^4 \\ &= \lambda^4 x^4 - 2\lambda^4 x y^3 - 5\lambda^4 y^4 \\ &= \lambda^4 (x^4 - 2xy^3 - 5y^4) \\ &= \lambda^4 f(x,y) \end{aligned}$$

$\therefore f$ is a homogeneous function of degree 4 #

HW ⊛ Function Cobb-Douglas. ($f : \mathbb{R}^n_+ \to \mathbb{R}$)

$$f(x_1, x_2, \dots, x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} \quad \alpha_i \text{ ใดๆ}$$

จงแสดงว่า ① ฟังก์ชัน Cobb-Douglas เป็นฟังก์ชันเอกพันธ์ ② หาว่าดีกรีเท่าไร

**Thm 1.9** Euler's Theorem for homogeneous function

If $f(x_1, \dots, x_n)$ is a homogeneous function of deg $m$, and the First order of partial derivative exist,

**Then** $x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \dots + x_n \frac{\partial f}{\partial x_n} = m f(x_1, \dots, x_n)$

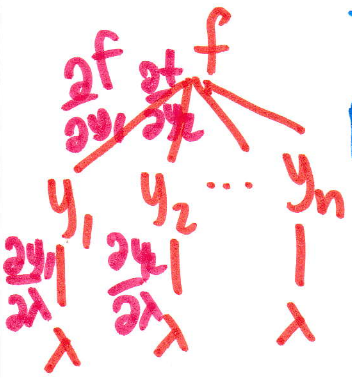
**Proof** Suppose that $f$ is a homogeneous function of deg. $m$.

Then $\forall \lambda > 0$,

$$f(\lambda x_1, \lambda x_2, \dots, \lambda x_n) = \lambda^m f(x_1, x_2, \dots, x_n)$$

Assume that $y_i = \lambda x_i$

$$f(y_1, y_2, \dots y_n) = \lambda^m f(x_1, \dots, x_n)$$




By chain rule, we have (Diff w.r.t. $\lambda$)

$$\frac{\partial f}{\partial y_1} \cdot \frac{\partial y_1}{\partial \lambda} + \frac{\partial f}{\partial y_2} \cdot \frac{\partial y_2}{\partial \lambda} + \dots + \frac{\partial f}{\partial y_n} \cdot \frac{\partial y_n}{\partial \lambda} = m \lambda^{m-1} f(x_1, \dots x_n)$$

From $\boxed{y_i = \lambda x_i,}$ $\frac{\partial y_i}{\partial \lambda} = x_i \quad \forall i = 1, \dots, n$

$$x_1 \frac{\partial f}{\partial y_1} + x_2 \frac{\partial f}{\partial y_2} + \dots + x_n \frac{\partial f}{\partial y_n} = m \lambda^{m-1} f(x_1, \dots, x_n)$$

From $y_i = \lambda x_i$; $\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial y_i} \cdot \frac{\partial y_i}{\partial x_i} \Rightarrow \frac{\partial f}{\partial x_i} = \lambda \frac{\partial f}{\partial y_i} \; \forall i$.

$$\frac{x_1}{\lambda} \frac{\partial f}{\partial x_1} + \frac{x_2}{\lambda} \frac{\partial f}{\partial x_2} + \dots + \frac{x_n}{\lambda} \frac{\partial f}{\partial x_n} = m \lambda^{m-1} f(x_1, \dots, x_n)$$

$$x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \dots + x_n \frac{\partial f}{\partial x_n} = m \lambda^{m} f(x_1, \dots x_n)$$

set $\lambda = 1$

$$\boxed{x_1 \frac{\partial f}{\partial x_1} + x_2 \frac{\partial f}{\partial x_2} + \dots + x_n \frac{\partial f}{\partial x_n} = m f(x_1, \dots, x_n).}$$

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