$$f(\lambda x_1, \lambda x_2, \ldots, \lambda x_n) = \lambda^m f(x_1, \ldots, x_n).$$

(Cobb-Douglas) $f(x_1, \ldots x_n) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$; $(f: \mathbb{R}_+^n \to \mathbb{R})$

Homogeneous function $\alpha_1 + \alpha_2 + \ldots + \alpha_n$ #

Ex 1.27 Let $f(x,y) = \sqrt{y^2 - x^2}$. show that

(1) f is homogeneous function

(2) f satisfies Thm. 1.9

$\left(x_1 \frac{\partial f}{\partial x_1} + \ldots + x_n \frac{\partial f}{\partial x_n} = m f(x_1, \ldots, x_n) \right)$

Proof (1) consider $f(\lambda x, \lambda y) = \sqrt{(\lambda y)^2 - (\lambda x)^2}$

$$= \sqrt{\lambda^2 (y^2 - x^2)}$$

$$= |\lambda| \sqrt{y^2 - x^2}$$

$$= \lambda \sqrt{y^2 - x^2} = \lambda f(x,y)$$

$\therefore f$ is homogeneous of degree 1.

(2) $\frac{\partial f}{\partial x} = \frac{(-2x)}{2\sqrt{y^2 - x^2}} = -\frac{x}{\sqrt{y^2 - x^2}}$

$\frac{\partial f}{\partial y} = \frac{2y}{2\sqrt{y^2 - x^2}} = \frac{y}{\sqrt{y^2 - x^2}}$

$\therefore x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = -\frac{x^2}{\sqrt{y^2 - x^2}} + \frac{y^2}{\sqrt{y^2 - x^2}}$

$$= \frac{y^2 - x^2}{\sqrt{y^2 - x^2}} = \sqrt{y^2 - x^2} = f(x,y)$$

#

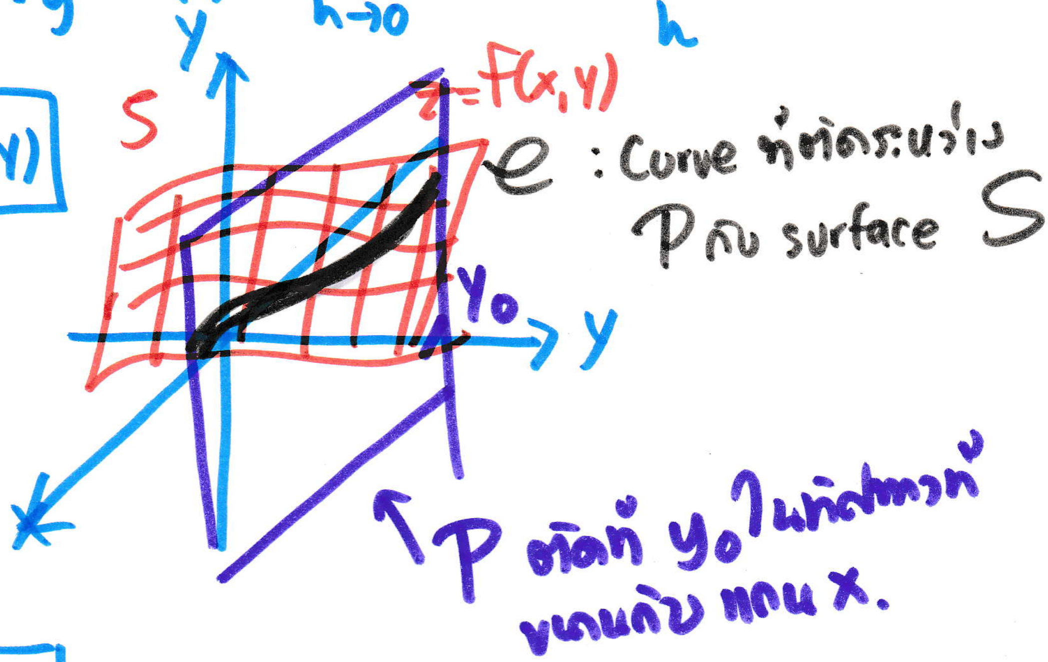
Ex 1.28 $f(x,y) = x^4 + 2xy^3 - 5y^4$.

1.7 Directional Derivatives.

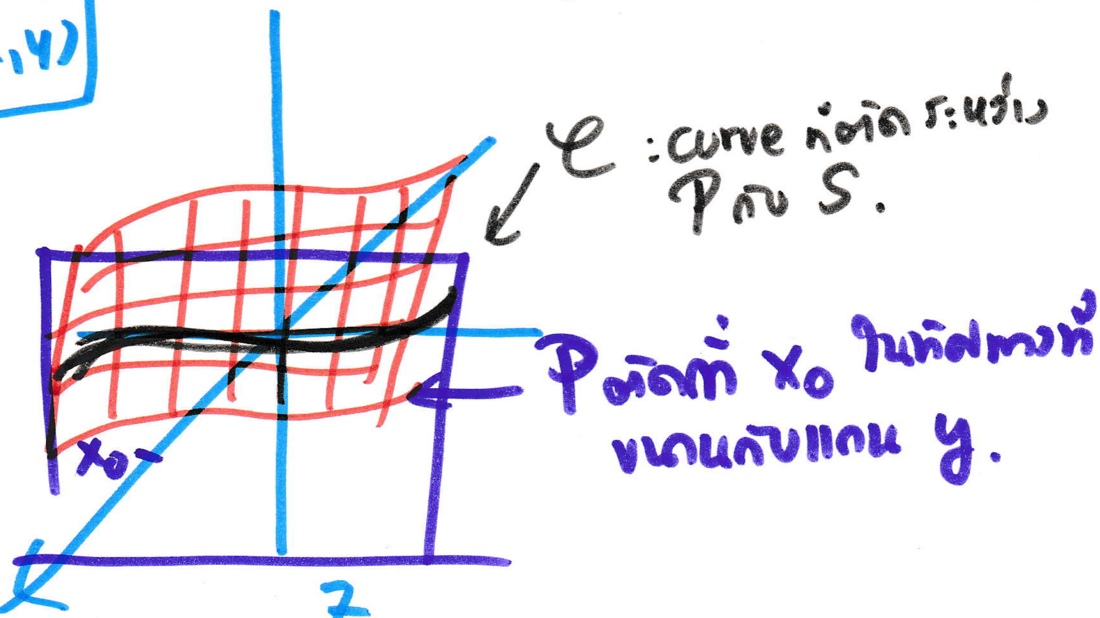
$$D_x f(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$D_y f(x,y) = \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$$

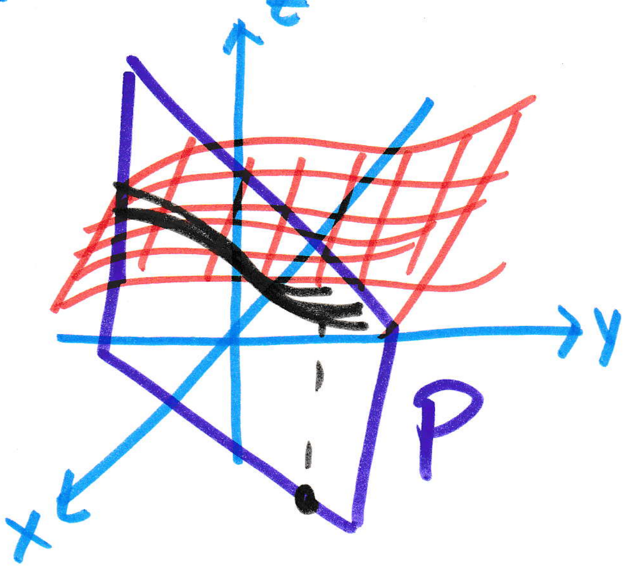
$\boxed{D_x f(x,y)}$

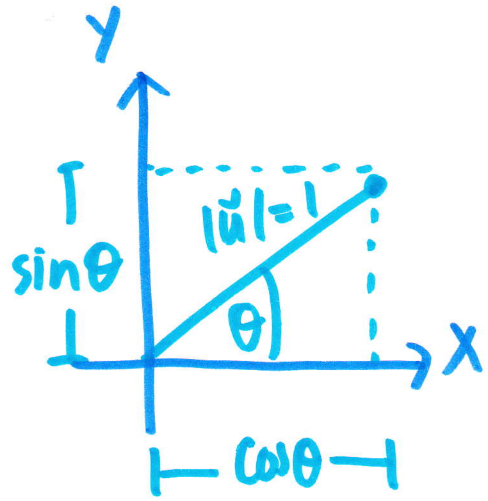


$\boxed{D_y f(x,y)}$



In general





Let f be a function of x, y.
$P(x,y)$ a point in the domain of f.
$\vec{u}$ a unit vector where the angle between $\vec{u}$ and x-axis is θ.

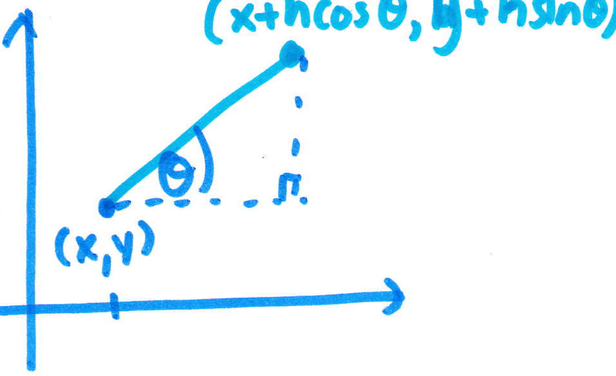
$$\vec{u} = (\cos\theta)\vec{i} + (\sin\theta)\vec{j}$$

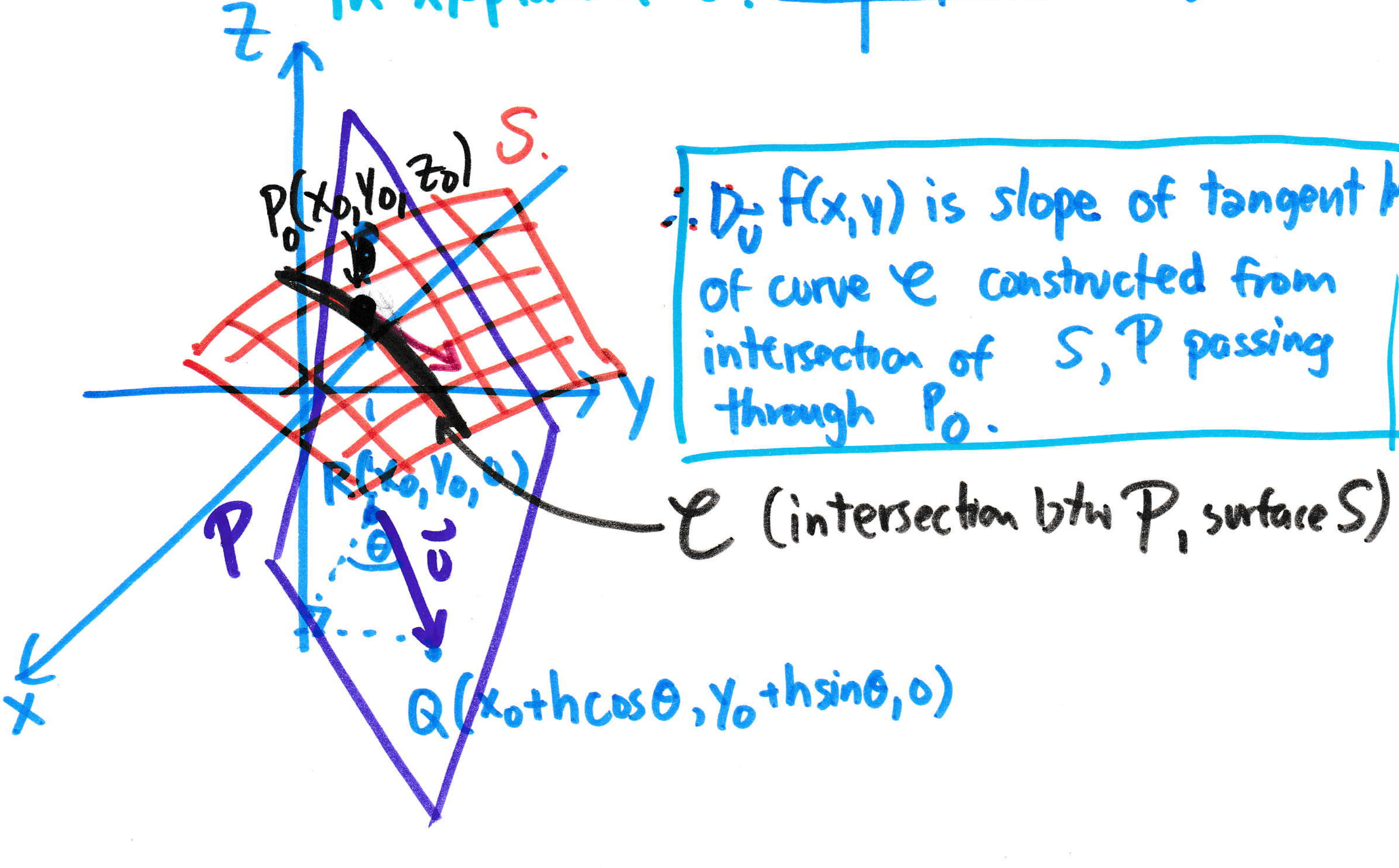
The <u>directional derivative</u> of f with the direction $\vec{u}$, is defined by

$$D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{\boxed{f(x+h\cos\theta,\, y+h\sin\theta)} - f(x,y)}{h}$$

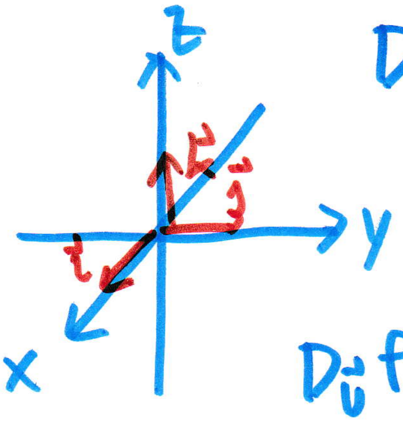
if limit exists.

Rate of changes of $f(x,y)$ w.r.t. the distance in xy-plane of $\vec{u}$.





$\therefore D_{\vec{u}} f(x,y)$ is slope of tangent l
of curve $\mathcal{C}$ constructed from
intersection of S, P passing
through P_0.



$$D_{\vec{u}}f(x,y)=\lim_{h\to 0}\frac{f(x+h\cos\theta,\,y+h\sin\theta)-f(x,y)}{h}$$

$\boxed{\text{เทียบ } D_x f(x,y)}$ เลือก $\vec{u}=\vec{i}.\ (\theta=0)$

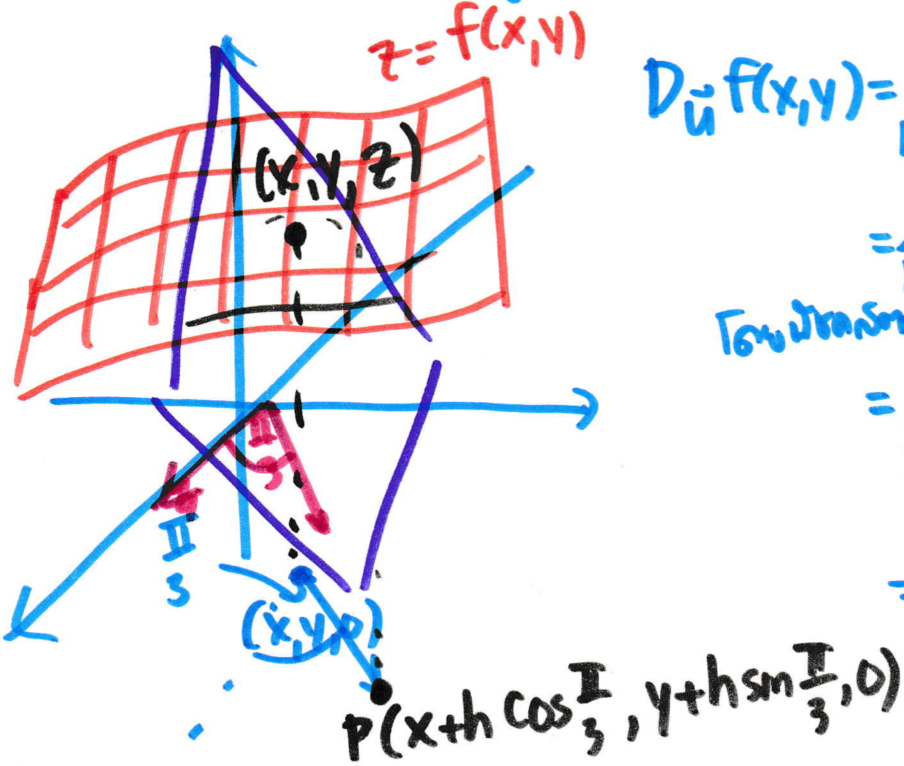
$$D_{\vec{u}}f(x,y)=\lim_{h\to 0}\frac{f(x+h,\,y)-f(x,y)}{h}=D_xf(x,y)$$

อยากได้ $\boxed{D_y f(x,y)}$ เลือก $\vec{u}=\vec{j}\ \ (\theta=\frac{\pi}{2})$

$$D_{\vec{u}}f(x,y)=\lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}.=D_yf(x,y).$$

Ex 1.29 Let $f(x,y)=x^2-3y$. $\vec{u}$ is a unit vector s.t. the angle btw. x-axis and $\vec{u}$ is $\frac{\pi}{3}$.

Find $D_{\vec{u}}f(x,y)$.



$$D_{\vec{u}}f(x,y)=\lim_{h\to 0}\frac{f(x+h\cos\frac{\pi}{3},\,y+h\sin\frac{\pi}{3})-f(x,y)}{h}$$

$$=\lim_{h\to 0}\frac{f(x+\frac{h}{2},\,y+\frac{h\sqrt{3}}{2})-f(x,y)}{h}$$

โดยปรกลรหุ้วแล้ว

$$=\lim_{h\to 0}\frac{x\cancel{h}+\frac{h^2}{4}-\frac{3\cancel{h}\sqrt{3}}{2}}{\cancel{h}}$$

$$=x-\frac{3\sqrt{3}}{2}\qquad\#$$

$A=a_1\vec{i}+a_2\vec{j}+a_3\vec{k}$, $B=b_1\vec{i}+b_2\vec{j}+b_3\vec{k}$

$\vec{A}\cdot\vec{B}=a_1b_1+a_2b_2+a_3b_3$

Gradient: $\nabla f(x,y)=f_x(x,y)\vec{i}+f_y(x,y)\vec{j}.$

Del operator: $\nabla=\frac{\partial}{\partial x}\vec{i}+\frac{\partial}{\partial y}\vec{j}.$

Thm 1.10 Let f be a differentiable funtion of x, y.

If $\vec{u} = \cos\theta\,\vec{i} + \sin\theta\,\vec{j}$, then

$$D_{\vec{u}} f(x,y) = \vec{u} \cdot \nabla f(x,y)$$

$$= \cos\theta\, f_x(x,y) + \sin\theta\, f_y(x,y).$$

Proof Let g be a function defined by

$$g(t) = f(\overset{r}{x + t\cos\theta}, \overset{s}{y + t\sin\theta})$$ (x, y, θ are regarded as constant.)

$$\therefore g'(0) = \lim_{h\to 0} \frac{g(0+h) - g(0)}{h}$$

$$= \lim_{h\to 0} \frac{f(x + h\cos\theta, y + h\sin\theta) - f(x,y)}{h}$$

$$= D_{\vec{u}} f(x,y).$$

Let $r = x + t\cos\theta$, $s = y + t\sin\theta$. Then $g(t) = f(r,s)$

$$g'(t) = \frac{\partial f}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial f}{\partial s}\cdot\frac{ds}{dt}$$

$$= \frac{\partial f}{\partial r}\cdot\cos\theta + \frac{\partial f}{\partial s}\cdot\sin\theta.$$

When $t = 0$; $r = x$, $s = y$

$$\therefore g'(0) = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

$$\therefore D_{\vec{u}} f(x,y) = \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta. \quad \#$$

Ex $f(x,y) = x^2 - 3y$.

$$\vec{u} = \cos\frac{\pi}{3}\vec{i} + \sin\frac{\pi}{3}\vec{j} = \frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}$$

$$\nabla f(x,y) = (2x)\vec{i} + (-3)\vec{j}$$

$$D_{\vec{u}} f(x,y) = \vec{u} \cdot \nabla f(x,y)$$

$$= \left(\frac{1}{2}\vec{i} + \frac{\sqrt{3}}{2}\vec{j}\right)\cdot\left(2x\vec{i} - 3\vec{j}\right) = x - \frac{3\sqrt{3}}{2} \quad \#$$

Ex Let $f(x,y)=\frac{x^2}{25}+\frac{y^2}{16}$

(1) Find $\nabla f(5,-4)$

(2) Find the rate of change of $f(x,y)$ with direction $\frac{\pi}{6}$ at $(5,-4)$.

Sol^n (1) หา $\nabla f(x,y)=f_x(x,y)\vec{i}+f_y(x,y)\vec{j}$.

$$=\frac{2x}{25}\vec{i}+\frac{2y}{16}\vec{j}.$$

$$\nabla f(5,-4)=\frac{2}{5}\vec{i}-\frac{1}{2}\vec{j}.$$

(2) หา $D_{\vec{U}}f(5,-4)$: หา ① $\vec{U}=\cos\left(\frac{\pi}{6}\right)\vec{i}+\sin\left(\frac{\pi}{6}\right)\vec{j}$

$$\vec{U}=\frac{\sqrt{3}}{2}\vec{i}+\frac{1}{2}\vec{j}$$

② $\nabla f(5,-4)$

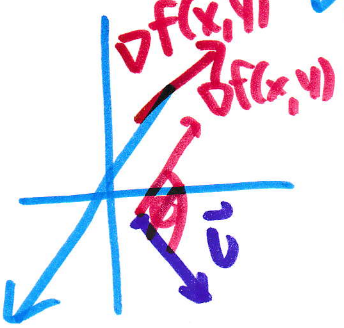
$$\therefore D_{\vec{U}}f(5,-4)=\left(\frac{\sqrt{3}}{2}\vec{i}+\frac{1}{2}\vec{j}\right)\cdot\left(\frac{2}{5}\vec{i}-\frac{1}{2}\vec{j}\right)$$

$$=\frac{\sqrt{3}}{5}-\frac{1}{4} \qquad \#.$$

Let ϕ be angle between $\vec{U}$ and $\nabla f(x,y)$, then

$$D_{\vec{U}}f(x,y)=\vec{U}\cdot\nabla f(x,y).$$

$$=|\vec{u}|\cdot|\nabla f(x,y)|\cdot\cos\phi$$

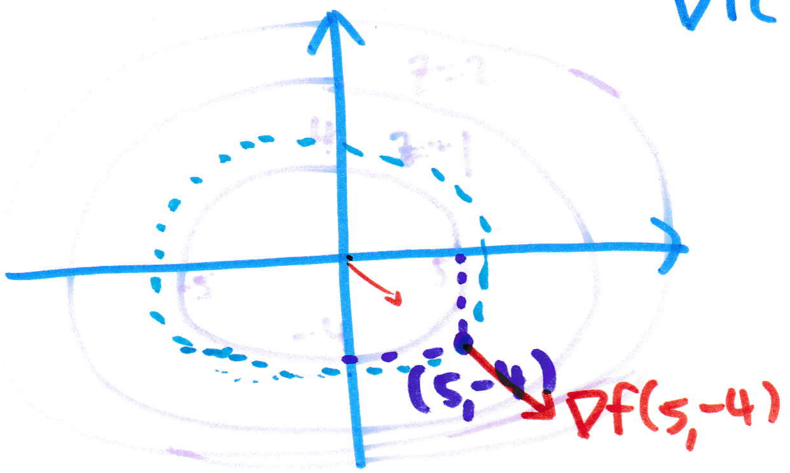


$\therefore$ Maximum of $D_{\vec{U}}f(x,y)$ occurs when $\phi=0$

Thus, higest rate of change is determined by $\nabla f(x,y)$.

$f(x,y)=\frac{x^2}{25}+\frac{y^2}{16}$

$\nabla f(5,-4)=\frac{2}{5}\vec{i}-\frac{1}{2}\vec{j}.$



<u>Thm</u> If $f(x,y)$ has continuous first partial derivatives in a domain containing the point (x_0,y_0), then at this point f will increase rapidly in the direction of its gradient $\nabla f(x_0,y_0)$

<u>Thm</u> Assume that $f(x,y)$ has continuous first partial derivatives in an open disk at (x_0,y_0), and that $\nabla f(x_0,y_0)\neq 0$, Then $\nabla f(x_0,y_0)$ is <u>normal</u> to the level curve of f through (x_0,y_0)