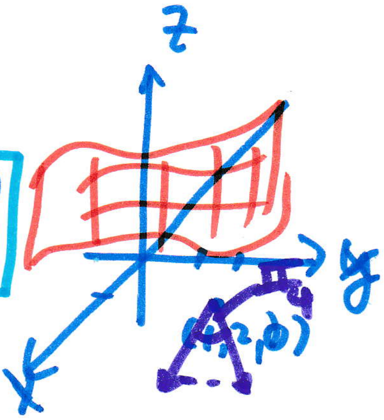
[illegible]

Ex Let $T(x,y) = 3x^2 + 2y^2$ be the temperature at (x,y) of a rectangular sheet on xy-plane. Find

① Rate of change of temperature at $(1,2)$ with direction $\frac{\pi}{4}$ with x-axis.

② Find the magnitude and direction of maximum rate of change at $(1,2)$

Sol^n Find $\vec{U} = \cos(\frac{\pi}{4})\vec{i} + \sin(\frac{\pi}{4})\vec{j}$



① $\Rightarrow$ $\boxed{D_{\vec{u}} f(x,y) = \lim_{h\to 0} \frac{f(x+h\cos\theta, y+h\sin\theta) - f(x,y)}{h}}$

$\Downarrow$

$$\boxed{D_{\vec{u}} f(x,y) = \vec{U} \cdot \nabla T(x,y)}$$

$$\nabla T(x,y) = \frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j}$$

$$= (6x)\vec{i} + (4y)\vec{j}$$

$$\therefore D_{\vec{u}} f(x,y) = \left(\cos(\tfrac{\pi}{4})\vec{i} + \sin(\tfrac{\pi}{4})\vec{j}\right) \cdot (6x\vec{i} + 4y\vec{j})$$

$$= \left(\frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}\right) \cdot (6x\vec{i} + 4y\vec{j})$$

$$= 3\sqrt{2}x + 2\sqrt{2}y$$

$$\therefore D_{\vec{u}} f(1,2) = (3\sqrt{2})(1) + (2\sqrt{2})(2)$$

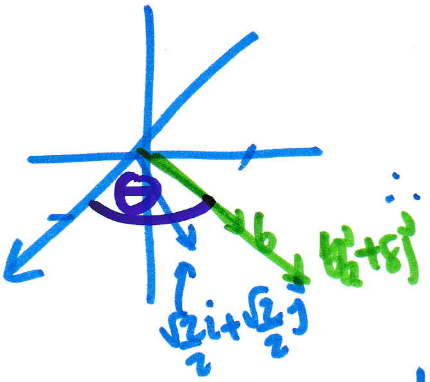
$$= 7\sqrt{2} \approx 9.90$$

Interpretation : At $(1,2)$, the temperature increase 9.90 unit/ distance of the direction of $\vec{U}$.

Maximum of $D_{\vec{u}}f(x,y) = \underbrace{|\vec{u}|}_{1} \cdot |\nabla T(x,y)| \cdot \underbrace{\cos\phi}_{1}$

$\therefore \nabla T(x,y) = \frac{\partial T}{\partial x}\vec{i} + \frac{\partial T}{\partial y}\vec{j}$

$= (6x)\vec{i} + (4y)\vec{j}.$

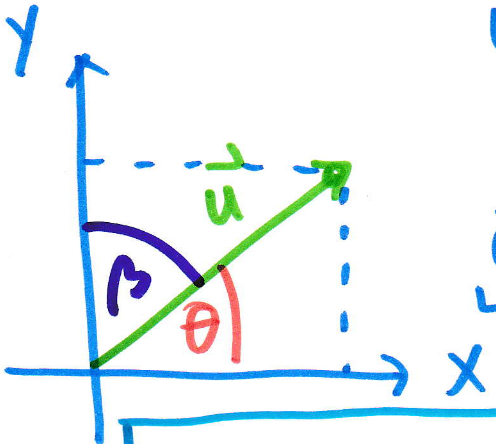


$\therefore \nabla T(1,2) = 6\vec{i} + 8\vec{j}$

$|\nabla T(1,2)| = \sqrt{(6)^2 + (8)^2} = 10.$

Since $\tan\theta = \frac{8}{6} \Rightarrow \theta = \arctan\left(\frac{4}{3}\right) \approx 0.93$ rad.

$\therefore$ The direction of maximum magnitude is 0.93 rad. measured from x-axis.



$\vec{u} = (\cos\theta)\vec{i} + (\sin\theta)\vec{j}.$

$\Downarrow$

$\underbrace{\vec{u} = (\cos\theta)\vec{i} + (\cos\beta)\vec{j}}_{\text{directional cosine.}}$

$$\boxed{D_{\vec{u}}f(x,y) = \lim_{h\to 0} \frac{f(x+h\cos\theta,\, y+h\cos\beta) - f(x,y)}{h}}$$

"directional derivative in term of directional cosine"

In general : $\vec{U} = (\cos\alpha)\vec{i} + (\cos\beta)\vec{j} + (\cos\gamma)\vec{k}$

$$\therefore D_{\vec{U}}f(x,y,z) = \lim_{h\to 0} \frac{f(x+h\cos\alpha,\, y+h\cos\beta,\, z+h\cos\gamma) - f(x,y,z)}{h}.$$

Thm 10.1 $D_{\vec{U}}f(x,y) = \vec{U} \cdot \nabla f(x,y)$

Ex 1.34 let $f(x,y,z) = e^{xyz} + 5x^2y - z^3$

ทิศทาง $3\vec{i} + 4\vec{j} - 12\vec{k}$ หา $D_{\vec{u}} f(1,-1,0)$

① หา $\vec{U}_1 = 3\vec{i} + 4\vec{j} - 12\vec{k}$ $\quad |\vec{u}_1| = \sqrt{3^2 + 4^2 + 12^2}$

② หา $\nabla f(x,y,z)$ $\quad = 13$

$$\vec{U} = \frac{\vec{u}_1}{|\vec{u}_1|} = \frac{3}{13}\vec{i} + \frac{4}{13}\vec{j} - \frac{12}{13}\vec{k}$$

$$\nabla f(x,y,z) = (yze^{xyz} + 10xy)\vec{i} + (xze^{xyz} + 5x^2)\vec{j} + (xye^{xyz} - 3z^2)\vec{k}$$

$$\nabla f(1,-1,0) = -10\vec{i} + 5\vec{j} - \vec{k}.$$

$\cancel{D_{\vec{u}} f(x,y) = \nabla \cdot \vec{U} \cdot \nabla}$

$$D_{\vec{u}} f(1,-1,0) = \vec{U} \cdot \nabla f(1,-1,0)$$

$$= \left(\frac{3}{13}\vec{i} + \frac{4}{13}\vec{j} - \frac{12}{13}\vec{k}\right) \cdot \left(-10\vec{i} + 5\vec{j} - \vec{k}\right)$$

$$= -\frac{30}{13} + \frac{20}{13} + \frac{12}{13} = +\frac{2}{13} \quad \#$$

1.8 Jacobian of Transformation

Let $F(u,v)$, and $G(u,v)$ be functions which are differentiable at a region. We define 「Jacobian Determinant」 or Jacobian of F and G. with respect to u, v by

$$\frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

For F, G, H functions of u, v, w.

$$\frac{\partial(F,G,H)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} & \frac{\partial F}{\partial w} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} & \frac{\partial G}{\partial w} \\ \frac{\partial H}{\partial u} & \frac{\partial H}{\partial v} & \frac{\partial H}{\partial w} \end{vmatrix} = \begin{vmatrix} F_u & F_v & F_w \\ G_u & G_v & G_w \\ H_u & H_v & H_w \end{vmatrix}$$

Thm 1.2 Let u, v be implicit functions of x, and y, determined by $F(x,y,u,v)=0$ and $G(x,y,u,v)=0$

$$\boxed{\frac{\partial u}{\partial x}} = -\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}, \quad \boxed{\frac{\partial u}{\partial y}} = -\frac{\frac{\partial(F,G)}{\partial(y,v)}}{\frac{\partial(F,G)}{\partial(u,v)}}$$

$$\boxed{\frac{\partial v}{\partial x}} = -\frac{\frac{\partial(F,G)}{\partial(u,x)}}{\frac{\partial(F,G)}{\partial(u,v)}}, \quad \boxed{\frac{\partial v}{\partial y}} = -\frac{\frac{\partial(F,G)}{\partial(u,y)}}{\frac{\partial(F,G)}{\partial(u,v)}}.$$

Proof Let $F(x,y,u,v)=0$, $G(x,y,u,v)=0$,

$u=u(x,y)$, $v=v(x,y)$.

$$\therefore dF=\frac{\partial F}{\partial x}dx+\frac{\partial F}{\partial y}dy+\frac{\partial F}{\partial u}\boxed{du}+\frac{\partial F}{\partial v}\boxed{dv}=0 \quad \text{—①}$$

$$dG=\frac{\partial G}{\partial x}dx+\frac{\partial G}{\partial y}dy+\frac{\partial G}{\partial u}\boxed{du}+\frac{\partial G}{\partial v}\boxed{dv}=0 \quad \text{—②}$$

By $u=u(x,y)$, $v=v(x,y)$

$$\boxed{du=\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy}, \quad \boxed{dv=\frac{\partial v}{\partial x}dx+\frac{\partial v}{\partial y}dy.}$$

Replace du, dv in ①, ②

$$0=dF=\left(\frac{\partial F}{\partial x}+\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\boxed{dx}+\left(\frac{\partial F}{\partial y}+\frac{\partial F}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial F}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\boxed{dy}$$

$$0=dG=\left(\frac{\partial G}{\partial x}+\frac{\partial G}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial G}{\partial v}\cdot\frac{\partial v}{\partial x}\right)\boxed{dx}+\left(\frac{\partial G}{\partial y}+\frac{\partial G}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial G}{\partial v}\cdot\frac{\partial v}{\partial y}\right)\boxed{dy}$$

Since dx, dy are L.I., we have

$$\boxed{\begin{aligned}F_x+F_u\cdot \boxed{u_x}+F_v\cdot\boxed{v_x}&=0\\ G_x+G_u\cdot\boxed{u_x}+G_v\cdot\boxed{v_x}&=0\end{aligned}}, \quad \boxed{\begin{aligned}F_y+F_u\cdot u_y+F_v\cdot v_y&=0\\ G_y+G_u\cdot u_y+G_v\cdot v_y&=0\end{aligned}}$$

By Cramer's rule, we solve for u_x, u_y, v_x, v_y

$$\boxed{\begin{bmatrix}F_u & F_v\\ G_u & G_v\end{bmatrix}\begin{bmatrix}u_x\\ v_x\end{bmatrix}=\begin{bmatrix}-F_x\\ -G_x\end{bmatrix}} \quad \boxed{\begin{bmatrix}F_u & F_v\\ G_u & G_v\end{bmatrix}\begin{bmatrix}u_y\\ v_y\end{bmatrix}=\begin{bmatrix}-F_y\\ -G_y\end{bmatrix}}$$

⋮

Ex let u, v be implicit functions of x, y, z s.t.

$$x^2+y^2+z^2-u^2+v^2-1 = 0 \qquad u=u(x,y,z)$$
$$x^2-y^2+z^2+u^2+2v^2-21 = 0 \qquad v=v(x,y,z)$$

Find $\frac{\partial u}{\partial x}$ at $x=1, y=1, z=2, u=3, v=2$

Sol^n: Let $F(x,y,z,u,v) = x^2+y^2+z^2-u^2+v^2-1 = 0$
$G(x,y,z,u,v) = x^2-y^2+z^2+u^2+2v^2-21 = 0$

$$\frac{\partial u}{\partial x} = -\frac{\frac{\partial(F,G)}{\partial(x,v)}}{\frac{\partial(F,G)}{\partial(u,v)}} = -\frac{\begin{vmatrix} F_x & F_v \\ G_x & G_v \end{vmatrix}}{\begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}}$$

$$= -\frac{\begin{vmatrix} 2x & 2v \\ 2x & 4v \end{vmatrix}}{\begin{vmatrix} -2u & 2v \\ 2u & 4v \end{vmatrix}}$$

$$= -\frac{(8vx-4vx)}{(-8uv-4uv)} = \frac{4xv}{12uv} = \frac{x}{3u}$$

at $x=1, y=1, z=2, u=3, v=2$, we have

$$\frac{\partial u}{\partial x} = \frac{1}{9} \qquad \#$$