$F(u,v)$, $G(u,v)$

$$J(u,v) = \frac{\partial(F,G)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial F}{\partial u} & \frac{\partial F}{\partial v} \\ \frac{\partial G}{\partial u} & \frac{\partial G}{\partial v} \end{vmatrix} = \begin{vmatrix} F_u & F_v \\ G_u & G_v \end{vmatrix}$$

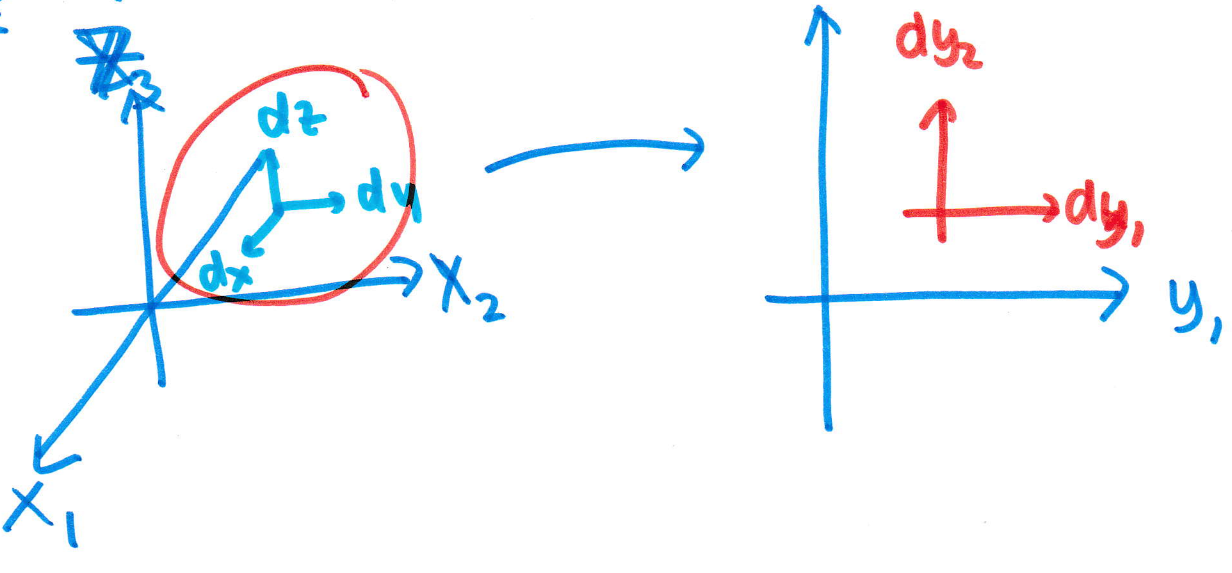
let
$$\left.\begin{aligned} y_1 &= f_1(x_1, \dots, x_n) \\ y_2 &= f_2(x_1, \dots, x_n) \\ &\vdots \\ y_m &= f(x_1, \dots, x_n) \end{aligned}\right\} \longrightarrow \begin{aligned} dy_1 &= \frac{\partial f_1}{\partial x_1} dx_1 + \dots + \frac{\partial f_1}{\partial x_n} dx_n \\ dy_2 &= \frac{\partial f_2}{\partial x_1} dx_1 + \dots + \frac{\partial f_2}{\partial x_n} dx_n \\ &\vdots \\ dy_m &= \frac{\partial f_m}{\partial x_1} dx_1 + \dots + \frac{\partial f_m}{\partial x_n} dx_n. \end{aligned}$$

$$\begin{bmatrix} dy_1 \\ dy_2 \\ \vdots \\ dy_m \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_m} \\ & & \vdots & \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_m} \end{bmatrix}}_{\text{Jacobian Matrix}} \begin{bmatrix} dx_1 \\ \vdots \\ dx_m \end{bmatrix}$$

* Transformation

(*) assign a point $(y_1, \dots, y_m)$ in $\mathbb{R}^m$ to $(x_1, \dots, x_n)$ in D_f.
Thus, they describe a mapping of D into $\mathbb{R}^m$.

Ex $D \subseteq \mathbb{R}^3 \longrightarrow \mathbb{R}^2$



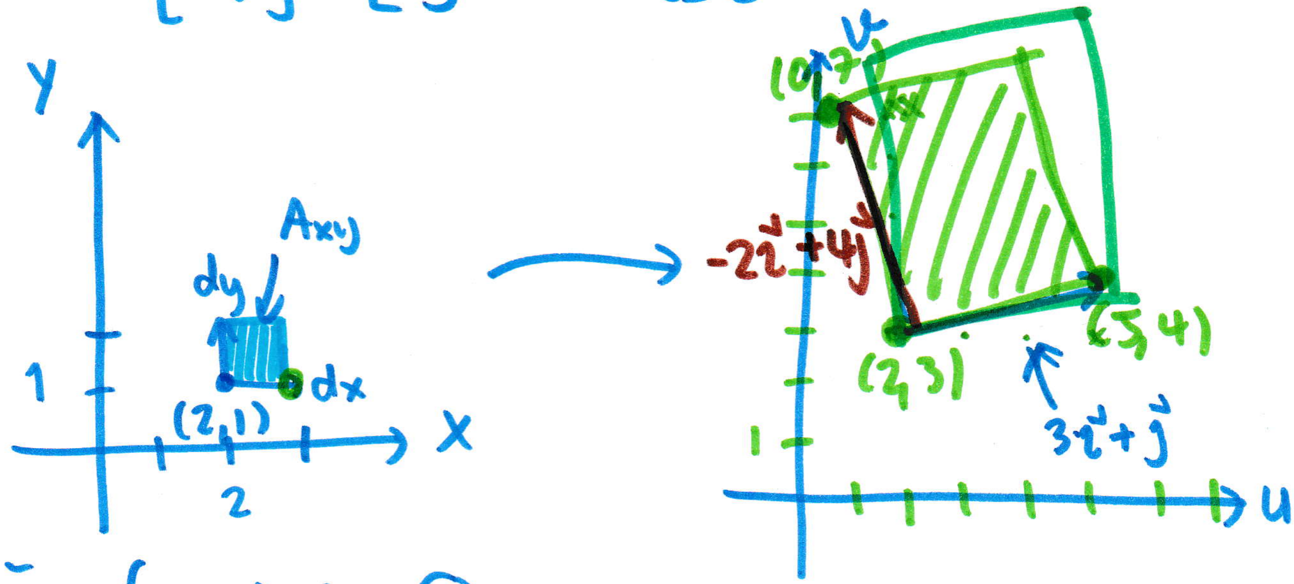
Ex $u = x^2 - xy$, $v = xy + y^2$. s.t. independent variable vector is (x, y).

Sol$^{\underline{n}}$

$$du = \frac{\partial}{\partial x}(x^2 - xy)\,dx + \frac{\partial}{\partial y}(x^2 - xy)\,dy$$

$$dv = \frac{\partial}{\partial x}(xy + y^2)\,dx + \frac{\partial}{\partial y}(xy + y^2)\,dy.$$

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} 2x - y & -x \\ y & x + 2y \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \quad \text{—— } (*)$$



ถ้าดู (2,1) จาก (*)

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix} \Rightarrow \begin{aligned} du &= 3dx - 2dy \\ dv &= dx + 4dy \end{aligned}$$

If $dy = 0$; $du = 3dx$, $dv = dx$
$dx = 0$; $du = -2dy$, $dv = 4dy$

Area of grid

$A_{xy} = 1$

Area of Grid

$$A_{uv} = \left| (3\vec{i} + \vec{j}) \times (-2\vec{i} + 4\vec{j}) \right|$$

$= 14$ ตร.หน่วย.

Theorem 1.14 Let x, y, u, v be related by the following relation

$$x = f(u,v) \quad , \quad y = g(u,v) \qquad \text{—} \; ①$$
$$u = F(x,y) \quad , \quad v = G(x,y) \qquad \text{—} \; ②$$

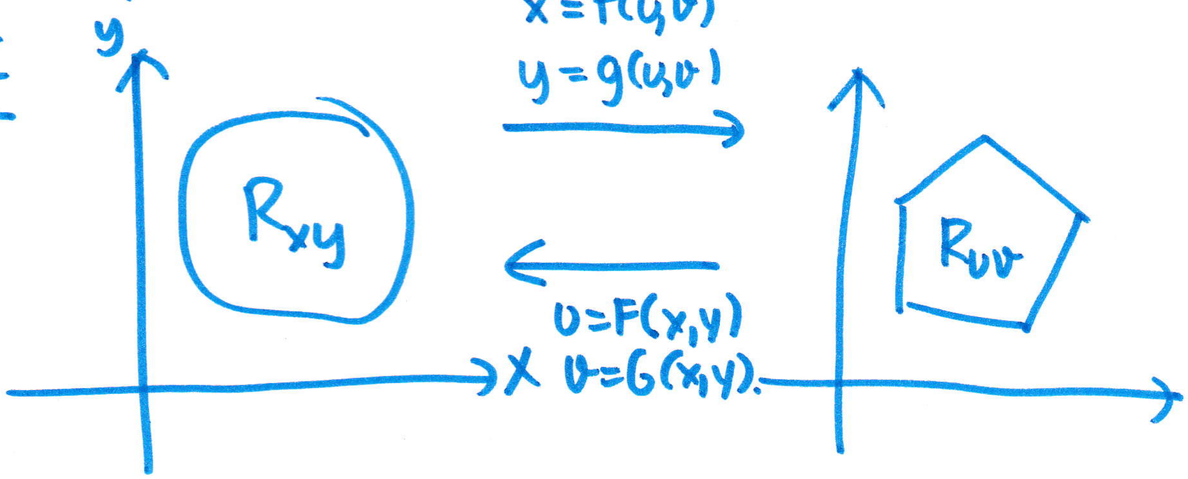
s.t. f, g, F, G, are continuous function and first order derivertives are continuous., then

$$A_{xy} = |J(u,v)| A_{uv} \text{ and } A_{uv} = |J(x,y)| A_{xy}.$$

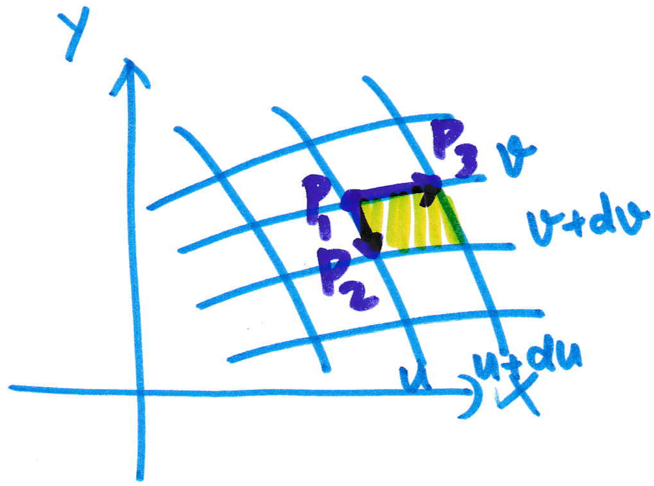
where $J(u,v) = \dfrac{\partial(x,y)}{\partial(u,v)}$, $J(x,y) = \dfrac{\partial(u,v)}{\partial(x,y)}$.

and A_{xy} ; A_{uv} are areas of regions in xy-plane and uv-plane, respectively, and satisfying $\boxed{1}$, $\boxed{2}$.

Sketch Proof



$\Rightarrow$



Ex 1.37 Let R_{xy} be a region enclosed by $x+y=6$, $x-y=2$ and $y=0$ on xy-plane.

The region R_{xy} is transformed by R_{uv} with

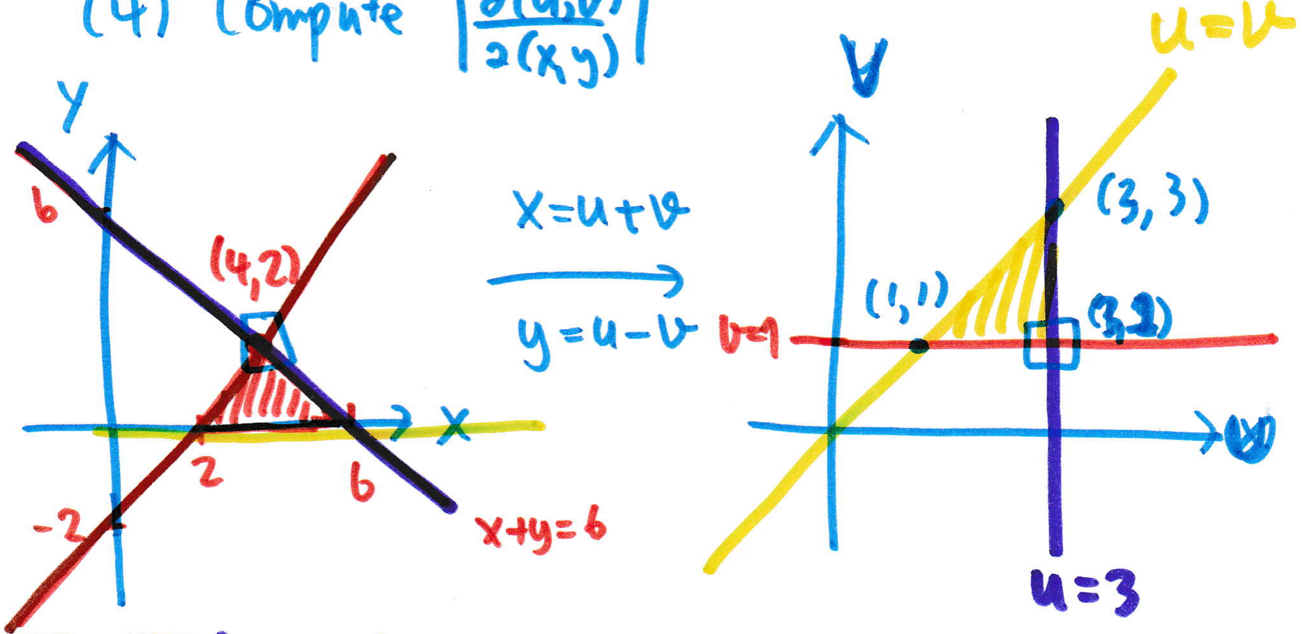
$$x = u+v, \quad y = u-v.$$

(1) Find R_{uv}

(2) Compute $\left|\frac{\partial(x,y)}{\partial(u,v)}\right|$

(3) Consider $\frac{A_{xy}}{A_{uv}}$

(4) Compute $\left|\frac{\partial(u,v)}{\partial(x,y)}\right|$



$\boxed{x+y=6} \Rightarrow (u+v)+(u-v)=6 \Rightarrow u=3$

$\boxed{x-y=2} \Rightarrow (u+v)-(u-v)=2 \Rightarrow v=1$

$\boxed{y=0} \Rightarrow 0 = u-v \Rightarrow u=v$

จุด (4,2) ถูกแปลงไปเป็นอะไร?

$\cancel{\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}}$ $\cancel{\frac{\partial(x,y)}{\partial(u,v)}}$ $\cancel{\frac{\partial(u,v)}{\partial(x,y)}}$

$\therefore \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 2 \end{bmatrix} = [$

(2) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$

$J(u,v)$

$$= \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

$$\therefore \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = 2$$

(3) $A_{xy} = (\tfrac{1}{2})(4)(2) = 4$

$A_{uv} = (\tfrac{1}{2})(2)(2) = 2$

(Thm 1.14) $A_{xy} = |J(u,v)| \cdot A_{uv}$

$$\therefore \frac{A_{xy}}{A_{uv}} = |J(u,v)| = 2$$

(4) $\left|\dfrac{\partial(u,v)}{\partial(x,y)}\right|$ จ $x = u+v$ ①, $y = u-v$ ②

$= \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix}$

①+②; $u = \frac{1}{2}(x+y)$

①−②; $v = \frac{1}{2}(x-y)$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$$\frac{A_{uv}}{A_{xy}} = \left|\frac{\partial(u,v)}{\partial(x,y)}\right| = \frac{1}{2}.$$

#

ดูว่า (4,2) ถูกแปลงไปหาใคร.

$$\left.\begin{aligned} du &= \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy \\ dv &= \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy \end{aligned}\right\} \begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

รู้ว่า $\underbrace{x = u+v, \quad y = u-v}$

$$u = \frac{1}{2}(x+y), \quad v = \frac{1}{2}(x-y)$$

$$\begin{bmatrix} du \\ dv \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} dx \\ dy \end{bmatrix}$$

$\therefore$ จุด (4,2)

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad \#$$

Thm 1.15 [Chain rule of transformation]

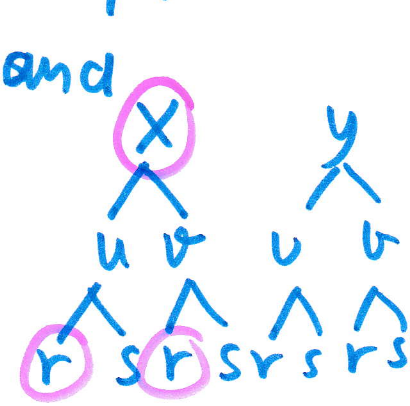
Suppose that $x = x(u,v)$, $y = y(u,v)$

$\boxed{u} = u(r,s)$, $\boxed{v} = v(r,s)$

which are differentiable function, and

$$x = F(r,s) = x(u(r,s), v(r,s))$$

$$y = G(r,s) = y(u(r,s), v(r,s))$$



Then $$\frac{\partial(x,y)}{\partial(r,s)} = \frac{\partial(x,y)}{\partial(u,v)} \cdot \frac{\partial(u,v)}{\partial(r,s)}$$

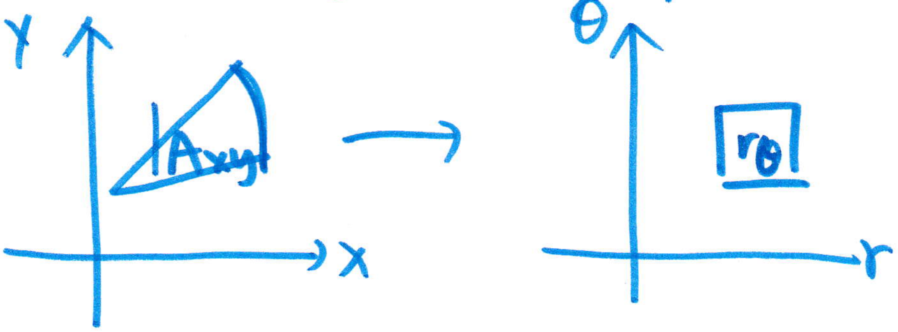
Sketch Proof $\dfrac{\partial(x,y)}{\partial(r,s)} = \begin{vmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial s} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial s} \end{vmatrix}$

idea นำ
$$\frac{\partial x}{\partial r} = \frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial r} + \frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial r}$$
$$\frac{\partial x}{\partial s} = \frac{\partial x}{\partial u}\cdot\frac{\partial u}{\partial s} + \frac{\partial x}{\partial v}\cdot\frac{\partial v}{\partial s}.$$
$$\frac{\partial y}{\partial r} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial r} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial r}$$
$$\frac{\partial y}{\partial s} = \frac{\partial y}{\partial u}\cdot\frac{\partial u}{\partial s} + \frac{\partial y}{\partial v}\cdot\frac{\partial v}{\partial s}.$$

(แทน แล้วจัดรูป ต่อ) . . . . #

Ex From rectangular coordinate (xy) and polar coordinate $(r\theta)$, the transformation is defined $x = r\cos\theta,\ y = r\sin\theta$

$\therefore$ Find relationship btw. A_{xy}, $A_{r\theta}$



$$A_{xy} = |J(r,\theta)|\cdot A_{r\theta}$$

$$\therefore |J(r,\theta)| = \frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$
$$= r(\cos^2\theta + \sin^2\theta) = r$$

$$\therefore A_{xy} = r A_{r\theta}.$$

#

Thm 1.16 Let x, y, z, and u, v, w, be related by

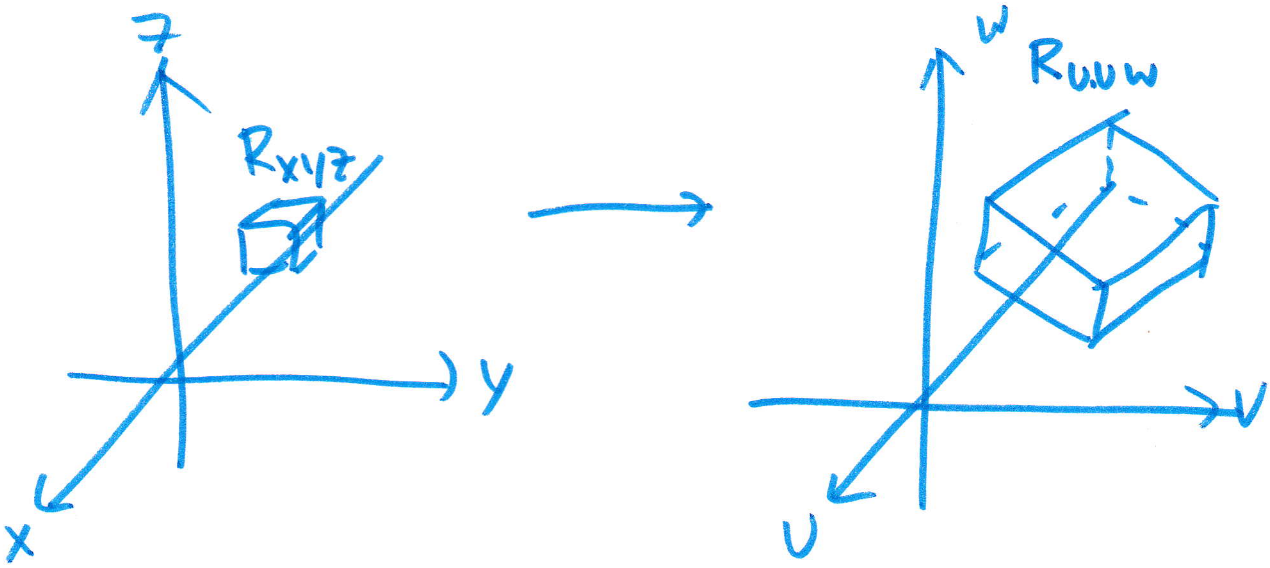
$$x = f(u,v,w),\ y = g(u,v,w),\ z = h(u,v,w)$$
$$u = F(x,y,z),\ v = G(x,y,z),\ w = H(x,y,z)$$

s.t. f, g, h, F, G, H are continuous and first par. der. exist and continuous in the domain.

Then $V_{xyz} = |J(u,v,w)| V_{uvw}$ and

$V_{uvw} = |J(x,y,z)| V_{xyz}$.

(Idea)



Ex Rectangular coordinate xyz
Cylindrical coordinate $r\theta z$

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

Prove that $\dfrac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$

Sol^n

$$\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= r$$

#