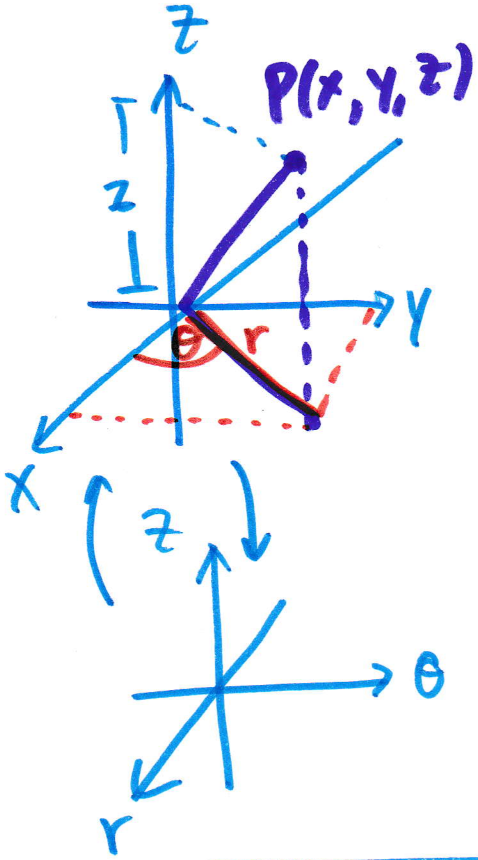
Cylindrical coordinate



$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \\ z &= z \end{aligned}\right\}$$

$(x, y, z) \longrightarrow (r, \theta, z)$

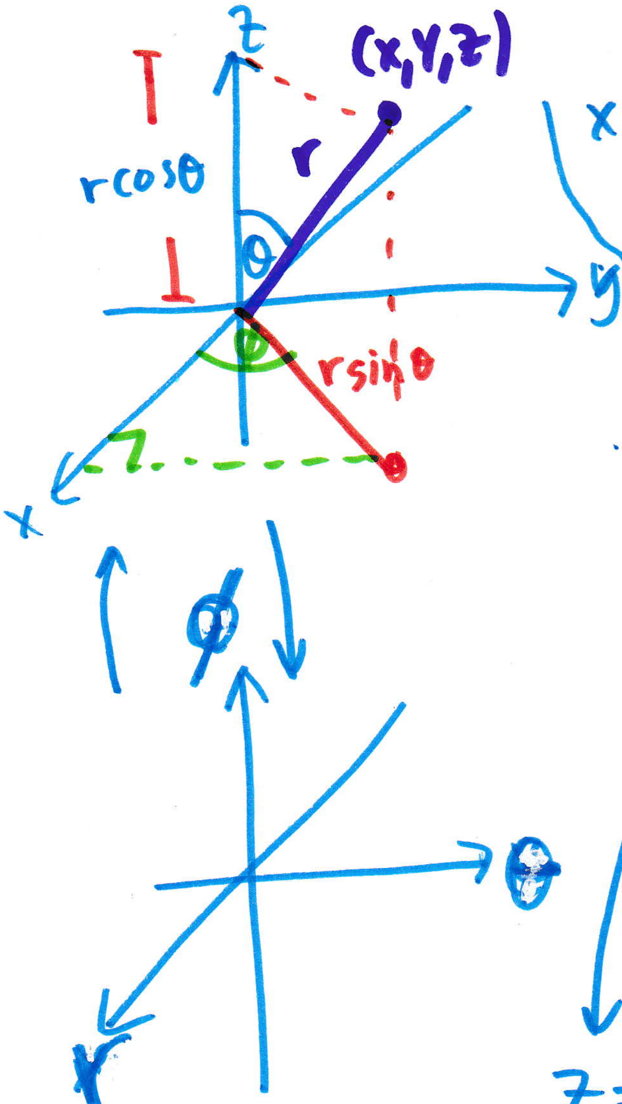
$r = \sqrt{x^2 + y^2}$

$\theta = \arctan\left(\frac{y}{x}\right)$

$z = z$

$(r, \theta, z) \rightarrow (x, y, z)$

Spherical Coordinate



$$\left.\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned}\right)$$

$(x, y, z) \longrightarrow (r, \theta, \phi)$

$r = \sqrt{x^2 + y^2 + z^2}$

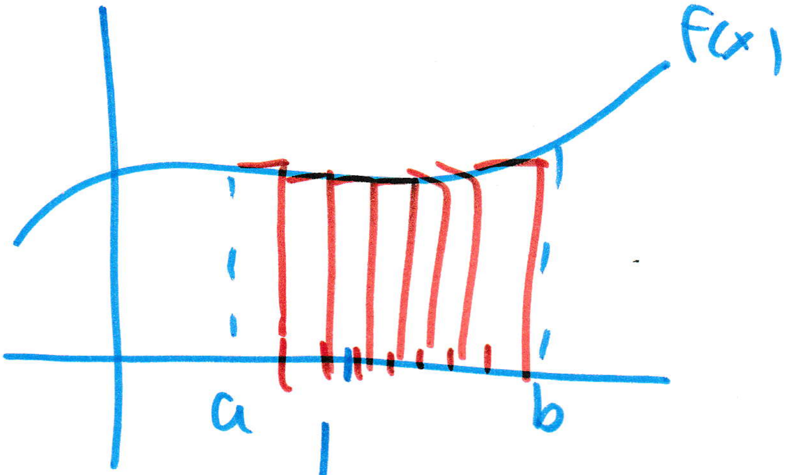
$\theta =$

$\phi \Rightarrow \frac{y}{x} = \tan\phi$

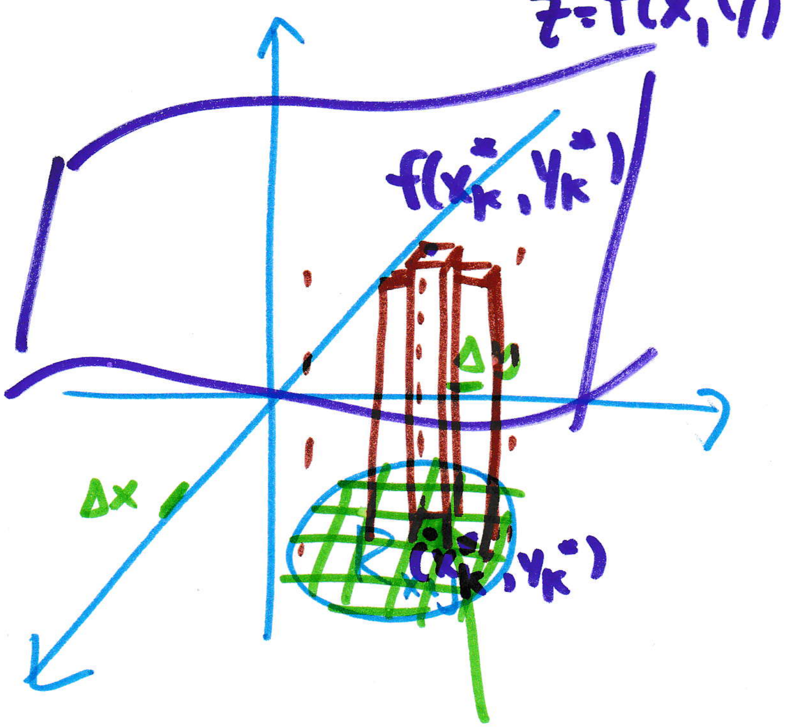
$\hookrightarrow \phi = \arctan\left(\frac{y}{x}\right)$

$z = \sqrt{x^2 + y^2 + z^2}\cos\theta$

$\cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}} \Rightarrow \theta = \arccos\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$



$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x_k$$



$$\iint_{R_{xy}} f(x,y)\,dy\,dx = \cancel{\lim} \lim_{\Delta y_k \to 0\ \Delta x_k \to \dots} \sum f(x_k^*, y_k^*)\Delta x_k \Delta y_k$$

$\Delta x \Delta y_k$

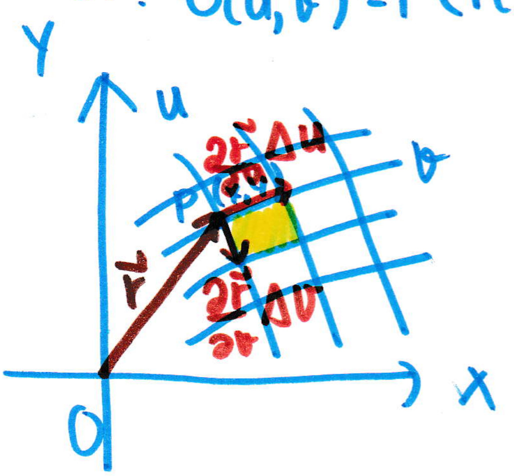
$$\iint_{R_{xy}} F(x,y)\,dx\,dy = \iint_{R_{uv}} G(u,v) \sim \,du\,dv$$

Thm let (u,v) be a curvilinear coordinate of a point in the plane s.t. $x=f(u,v)$, $y=g(u,v)$. which maps (x,y) in xy-plane to (u,v) in uv-plane. Such that R_{xy} is mapped to R_{uv}. Hence

$$\iint_{R_{xy}} F(x,y)\,dx\,dy = \iint_{R_{uv}} G(u,v)\,|J(u,v)|\,du\,dv.$$

s.t. $G(u,v) = F(f(u,v), g(u,v))$.

Proof



let P be a point with xy-coordinate (x,y) corresponds to (u,v). s.t.

$$x=f(u,v),\quad y=g(u,v)$$

Consider $\vec{r}$ from O to P.

$$\vec{r} = x\vec{i} + y\vec{j}.$$

$$\therefore \vec{r} = f(u,v)\vec{i} + g(u,v)\vec{j}.$$

Then the tangent vector at $u=c_1$, $v=c_2$ s.t. c_1, c_2 are constant, is, $\frac{\partial \vec{r}}{\partial v}$, $\frac{\partial \vec{r}}{\partial u}$

$$\Delta r = \left|\frac{\partial \vec{r}}{\partial u}\Delta u \times \frac{\partial \vec{r}}{\partial v}\Delta v\right| = \left|\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\right| \Delta u \Delta v$$

$$\therefore \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & 0 \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix}\vec{k} = \underbrace{\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}}_{J(u,v)}\vec{k}$$

$$= \frac{\partial(x,y)}{\partial(u,v)} = J(u,v)\vec{k}$$

Hence, $\left|\frac{\partial \vec{r}}{\partial u}\times\frac{\partial \vec{r}}{\partial v}\right| = |J(u,v)|\cdot \Delta u \Delta v$.

$$\left[\ \overset{dxdy}{A_{xy}} = |J(u,v)|\cdot \overset{dudv}{A_{uv}}\ \right]$$

$$\underbrace{\sum F(x,y)\,\Delta x \Delta y}_{\text{sum in xy.}} = \sum \underbrace{F(f(u,v),g(u,v))}_{\text{Set } G(u,v)}\,|J(u,v)|\cdot du\,dv$$

Take limit $\Delta u \to 0$, $\Delta v \to 0$, we have

$$\iint_{R_{uv}} G(u,v)\,J(u,v)\,du\,dv$$

s.t. $G(u,v) = F(f(u,v),g(u,v))$ #

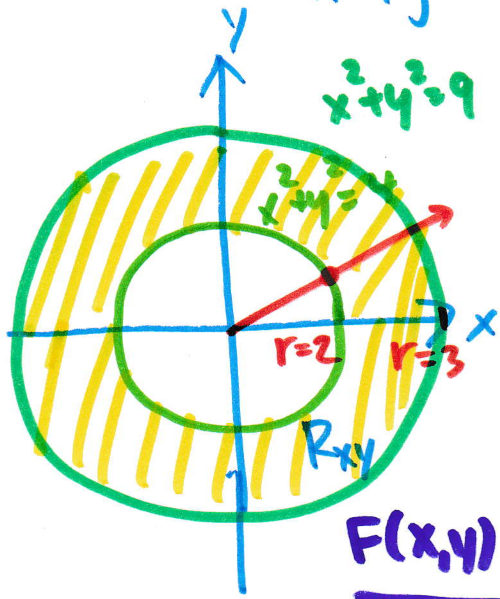
CAL2 $\quad \iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R_{r\theta}} g(r,\theta)\,r\,dr\,d\theta$

$\downarrow$

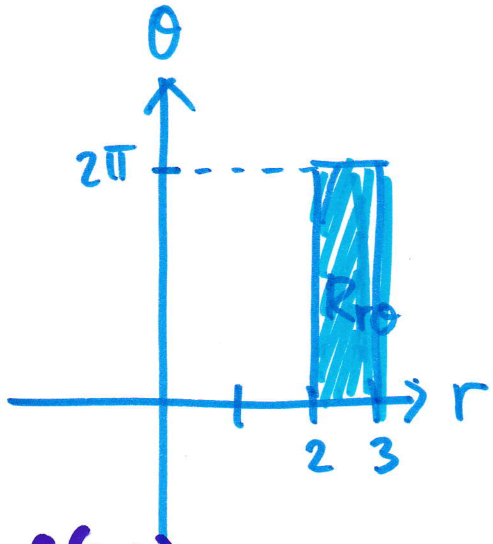
แปลง $x = r\cos\theta$

$y = r\sin\theta$

Ex $\iint_{R_{xy}} \sqrt{x^2+y^2}\,dx\,dy$ s.t. R_{xy} is enclosed by $x^2+y^2=4$ and $x^2+y^2=9$



$x = r\cos\theta$
$y = r\sin\theta$



$$\therefore \iint_{R_{xy}} \overbrace{\boxed{\sqrt{x^2+y^2}}}^{F(x,y)}\,dx\,dy = \iint_{R_{r\theta}} \overbrace{\boxed{r}}^{G(r,\theta)}\,|J(r,\theta)|\,dr\,d\theta$$

$$\sqrt{x^2+y^2} = \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} = r.$$

$$\therefore J(r,\theta) = \begin{vmatrix} x_r & x_\theta \\ y_r & y_\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta$$

$$= r$$

$$\therefore \iint_{R_{xy}} \sqrt{x^2+y^2}\,dx\,dy = \int_0^{2\pi}\int_2^3 (r)(r)\,dr\,d\theta = \int_0^{2\pi}\int_2^3 r^2\,dr\,d\theta$$

$$= \int_0^{2\pi} \frac{r^3}{3}\Big|_2^3\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{27}{3} - \frac{8}{3}\right) d\theta = \int_0^{2\pi} \frac{19}{3}\,d\theta = \frac{19}{3}\theta\Big|_0^{2\pi}$$

$$= \frac{38}{3}\pi$$

#.

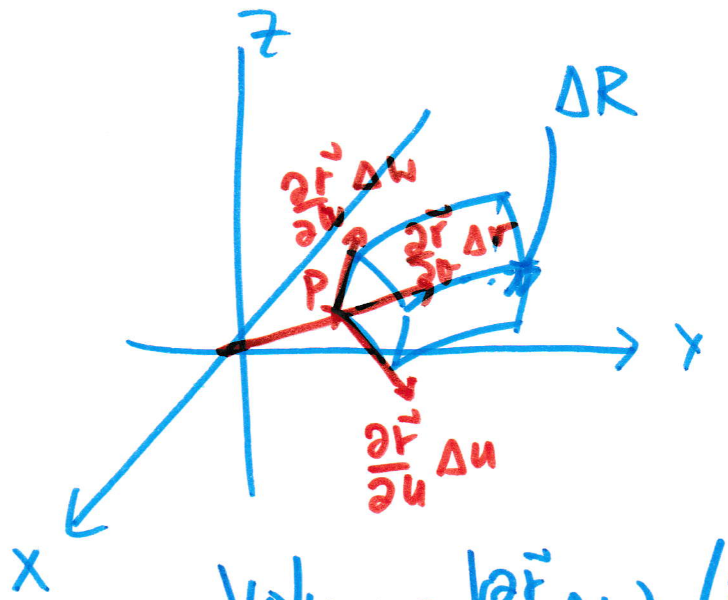
Thm 1.18 Let (u,v,w) be a curvilinear coordinate satisfying the equation

$$x = f(u,v,w), \quad y = g(u,v,w), \quad z = h(u,v,w).$$

Then
$$\iiint_{R_{xyz}} F(x,y,z)\,dx\,dy\,dz = \iiint_{R_{uvw}} H(u,v,w)\,|J(u,v,w)|\,du\,dv\,dw$$

s.t. $H(u,v,w) = F(f(u,v,w), g(u,v,w), h(u,v,w))$.

Idea



$$\text{Volume} = \left|\left(\frac{\partial \vec{r}}{\partial u}\Delta u\right)\cdot\left(\frac{\partial \vec{r}}{\partial v}\Delta v \times \frac{\partial \vec{r}}{\partial w}\Delta w\right)\right|$$

Ex $\displaystyle\iiint_{R_{xyz}} F(x,y,z)\,dx\,dy\,dz$

① Cylindrical coordinate
② spherical coordinate

① $x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$\iiint_{R_{xyz}} F(x,y,z)\,dx\,dy\,dz = \iiint_{R_{r\theta z}} G(r,\theta,z)\,\underline{|J(r,\theta,z)|}\,dr\,d\theta\,dz$$

$$\therefore J(r,\theta,z) = \begin{vmatrix} x_r & x_\theta & x_z \\ y_r & y_\theta & y_z \\ z_r & z_\theta & z_z \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

$$\therefore \iiint_{R_{xyz}} F(x,y,z)\,dx\,dy\,dz = \iiint_{R_{r\theta z}} F(r\cos\theta, r\sin\theta, z)\,r\,dr\,d\theta\,dz.$$

② $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

$$\iiint_{R_{xyz}} F(x,y,z)\,dx\,dy\,dz = \iiint_{R_{r\theta\phi}} G(r,\theta,\phi)\,|J(r,\theta,\phi)|\,dr\,d\theta\,d\phi$$

$$\therefore J(r,\theta,\phi) = \begin{vmatrix} x_r & x_\theta & x_\phi \\ y_r & y_\theta & y_\phi \\ z_r & z_\theta & z_\phi \end{vmatrix} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$\left(0 + r^2\cos^2\theta\sin\theta\cos^2\phi + \underline{r^2\sin^3\theta\sin^2\phi}\right) - \left(-r^2\cos^2\theta\sin\theta\sin^2\phi - \underline{r^2\sin^3\theta\cos^2\phi} + 0\right) = r^2\sin\theta$$

$r^2\sin^3\theta$

$$= r^2\cos^2\theta\sin\theta\cos^2\phi + r^2\cos^2\theta\sin\theta\sin^2\phi + r^2\sin^3\theta$$

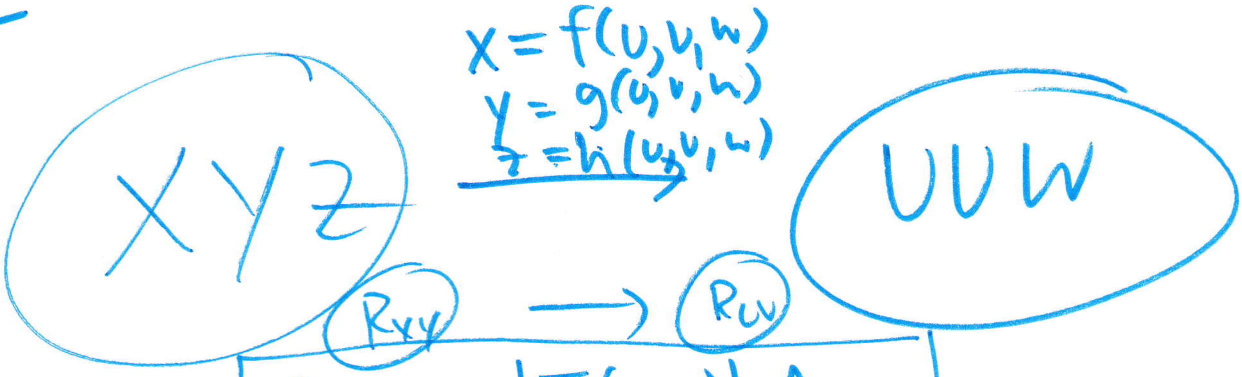
$$= r^2\sin\theta\left(\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta\right)$$

$$= r^2\sin\theta\left(\cos^2\theta\,\cancel{(\cos^2\phi+\sin^2\phi)} + \sin^2\theta\right)$$

$$\therefore \iiint_{R_{xyz}} F(x,y,z)\,dx\,dy\,dz = \iiint_{R_{r\theta\phi}} G(r,\theta,\phi)\,\underline{\underline{r^2 \sin\theta}}\,dr\,d\theta\,d\phi$$

$$\text{s.t. } G(r,\theta,\phi) = F(r\sin\theta\cos\phi,\ r\sin\theta\sin\phi,\ r\cos\theta) \ \#$$

Saroob



$$\boxed{A_{xy} = |J(u,v)| \cdot A_{uv}.}$$

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