1.10 Higher-order Partial Derivative

Let f be a function of two variables x and y.

$D_x f$, $D_y f$ are also function of two varibles.

$$f \to x, \; y$$

$$\frac{\partial f}{\partial x} \to \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \; \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

$$\frac{\partial f}{\partial y} \to \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right), \; \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)$$

- If the partial of $D_x f$, $D_y f$ exist,
  it is s.t.b. 「second order partial derivative」
- $D_y(D_x f) = D_{xy} f = f_{xy} = \dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$

■ เริ่มก่อน

■ ตามหลัง

แทนทุกตัวของฟังก์ชัน f_x (ของ)

- $f_{xy}(x,y) = \lim\limits_{\Delta y \to 0} \dfrac{f_x(x, y+\Delta y) - f_x(x,y)}{\Delta y}$
- $f_{xx}(x,y) = \lim\limits_{\Delta x \to 0} \dfrac{f_x(x+\Delta x, y) - f_x(x,y)}{\Delta x}$
- $f_{yx}(x,y) = \lim\limits_{\Delta x \to 0} \dfrac{f_y(x+\Delta x, y) - f_y(x,y)}{\Delta x}$
- $f_{yy}(x,y) = \lim\limits_{\Delta y \to 0} \dfrac{f_y(x, y+\Delta y) - f_y(x,y)}{\Delta y}$

$$\boxed{f_x(x,y) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}}$$

Ex $D_{xxy} f = f_{xxy} = \dfrac{\partial^3 f}{\partial y \partial x \partial x} = \dfrac{\partial^3 f}{\partial y \partial x^2}$

Ex let $f(x,y) = e^{-x} \sin 2y + x^2 + 4y^2$

จาก $f_x = -e^{-x} \sin 2y + 2x$

$f_y = 2e^{-x} \cos 2y + 8y$

$f_{xx} = e^{-x} \sin 2y + 2$

$f_{yy} = -4e^{-x} \sin 2y + 8$

$f_{yx} = -2e^{-x} \cos 2y$

$f_{xy} = -2e^{-x} \cos 2y$

$f_{yx} = f_{xy}$

Ex Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$ if f is defined by

$$f(x,y)=\begin{cases} xy\left(\frac{x^2-y^2}{x^2+y^2}\right) & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0)\end{cases}$$

We did it! $f_x(0,y) = -y$, $f_y(x,0) = x$

Then

$$f_{xy}(0,0) = \lim_{\Delta y\to 0} \frac{f_x(0,0+\Delta y)-f_x(0,0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{-\Delta y - 0}{\Delta y} = -1$$

$$f_{yx}(0,0) = \lim_{\Delta x\to 0} \frac{f_y(0+\Delta x,0)-f_y(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{\Delta x - 0}{\Delta x} = 1$$

$\therefore \quad f_{xy}(0,0) \neq f_{yx}(0,0)$

$$f_x(x,y)=\begin{cases} \frac{y(x^4+4x^2y^2-y^4)}{(x^2+y^2)^2} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0)\end{cases}$$

$$f_y(x,y)=\begin{cases} \frac{x(x^4-4x^2y^2-xy^4)}{(x^2+y^2)^2} & ; (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0)\end{cases}$$

Let $S\subseteq D_f$

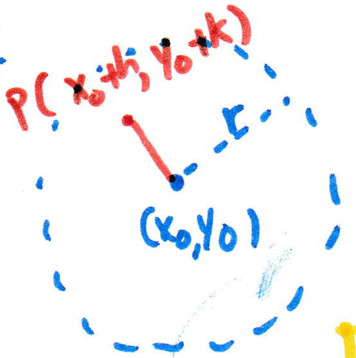
- $f\in C^n(S)$ if $\frac{\partial^n f}{\partial x^n}, \frac{\partial^n f}{\partial x^{n-1}\partial y}, \dots, \frac{\partial f^n}{\partial y^n}$ are cont.
- $f\in C^\infty(S)$ if $f\in C^n(S)$ $\forall n\in\mathbb{Z}^+$

Ex $f(x,y)=e^{xy}$; $f\in C^\infty(S)$

Theorem 1.19 Let f be a function of two variables x and y which exist in $N((x_0,y_0);r)$ and f_x, f_y, f_{yx}, f_{xy} exist in $N((x_0,y_0);r)$.

If f_{xy} and f_{yx} are continuous in $N((x_0,y_0);r)$, then
$$f_{xy}(x_0,y_0) = f_{yx}(x_0,y_0).$$

Proof



Let $P(x_0+h, y_0+k) \in N((x_0,y_0),r)$, and let
$$\phi(x) = f(x, y_0+k) - f(x, y_0)$$
$$\therefore \phi(x_0+h) = f(x_0+h, y_0+k) - f(x_0+h, y_0)$$
$$\therefore \boxed{\frac{\partial}{\partial x}\phi(x_0+h)} = \frac{\partial}{\partial x} f(x_0+h, y_0+k) - \frac{\partial}{\partial x} f(x_0+h, y_0) \quad \text{—(1)}$$

By MVT,
$$\phi(x_0+h) - \phi(x_0) = h\frac{\partial}{\partial x}\phi(\xi) = h\boxed{\frac{\partial}{\partial x}\phi(x_0+\theta_1 h)} \quad \text{—(2)}$$
s.t. $x_0 < \xi = x_0+\theta h < x_0+h \Rightarrow 0 < \theta_1 < 1$.

Put (1) in (2);
$$\therefore \phi(x_0+h) - \phi(x_0) = h\left[\frac{\partial}{\partial x} f(x_0+\theta_1 h, y_0+k) - \frac{\partial}{\partial x} f(x_0+\theta_1 h, y_0)\right].$$
$$\text{or } \phi(x_0+h) - \phi(x_0) = h\left[\boxed{f_x(x_0+\theta_1 h, y_0+k) - f_x(x_0+\theta_1 h, y_0)}\right]. \quad \text{—(3)}$$

By MVT, we have
$$\boxed{f_x(x_0+\theta_1 h, y_0+k) - f_x(x_0+\theta_1 h, y_0)} = k\frac{\partial}{\partial y} f_x(x_0+\theta_1 h, y_0+\theta_2 k) \quad \text{—(4)}$$
s.t. $0 < \theta_2 < 1$.

Put (4) in (3);
$$\phi(x_0+h) - \phi(x_0) = hk\left[\frac{\partial}{\partial y} f_x(x_0+\theta_1 h, y_0+\theta_2 k)\right]$$
$$\phi(x_0+h) - \phi(x_0) = hk f_{xy}(x_0+\theta_1 h, y_0+\theta_2 k) \quad \text{—(5)}$$

Since f_{xy} is cont. and exists in $N((x_0,y_0);r)$,

$$f_{xy}(x_0+\theta_1 h, y_0+\theta_2 k) = f_{xy}(x_0,y_0)+\varepsilon \quad —⑥$$

~~when~~ s.t. $\varepsilon \to 0$ where $(h,k)\to(0,0)$.

Put ⑥ in ⑤

$$\therefore \phi(x_0+h)-\phi(x_0) = hk[f_{xy}(x_0,y_0)+\varepsilon]$$

$$\frac{f(x_0+h, y_0+k)-f(x_0+h,y_0)}{k} - \frac{f(x_0,y_0+k)-f(x_0,y_0)}{k} = h[f_{xy}(x_0,y_0)+\varepsilon]$$

Take $k\to 0$

$$f_y(x_0+h,y_0) - f_y(x_0,y_0) = h[f_{xy}(x_0,y_0)+\varepsilon]$$

$$\therefore \frac{f_y(x_0+h,y_0)-f_y(x_0,y_0)}{h} = f_{xy}(x_0,y_0)+\varepsilon$$

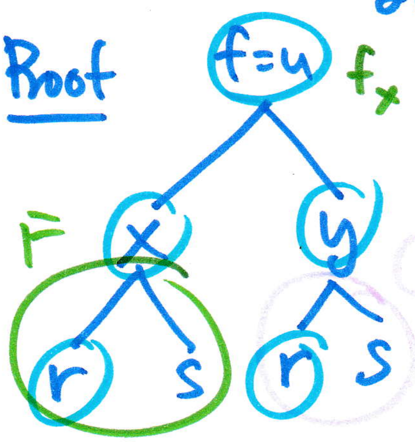
Take $h\to 0$

$$f_{yx}(x_0,y_0) = f_{xy}(x_0,y_0) \qquad \#$$

Ex Let $u=f(x,y)$, $x=F(r,s)$, $y=G(r,s)$. Assume that $f_{xy}=f_{yx}$. Prove that

$$\frac{\partial^2 u}{\partial r^2} = f_{xx}F_r^2 + 2f_{xy}F_rG_r + f_{yy}G_r^2 + f_xF_{rr} + f_yG_{rr}.$$

Proof

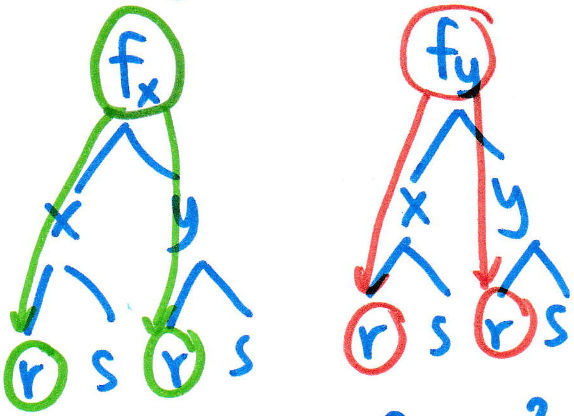


$$\frac{\partial u}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$\boxed{\frac{\partial u}{\partial r} = f_x(x,y)\cdot F_r(r,s) + f_y(x,y)\cdot G_r(r,s)}$$

$$\frac{\partial^2 u}{\partial r^2} = \left(f_x\frac{\partial F_r}{\partial r} + F_r\boxed{\frac{\partial f_x}{\partial r}}\right) + \left(f_y\frac{\partial G_r}{\partial r} + G_r\frac{\partial f_y}{\partial r}\right)$$

$$\therefore \frac{\partial^2 v}{\partial r^2} = f_x F_{rr} + F_r\left(f_{xx} \cdot F_r + f_{xy} \cdot G_r\right) + f_y G_{rr} + G_r\left(f_{yx} F_r + f_{yy} G_r\right).$$



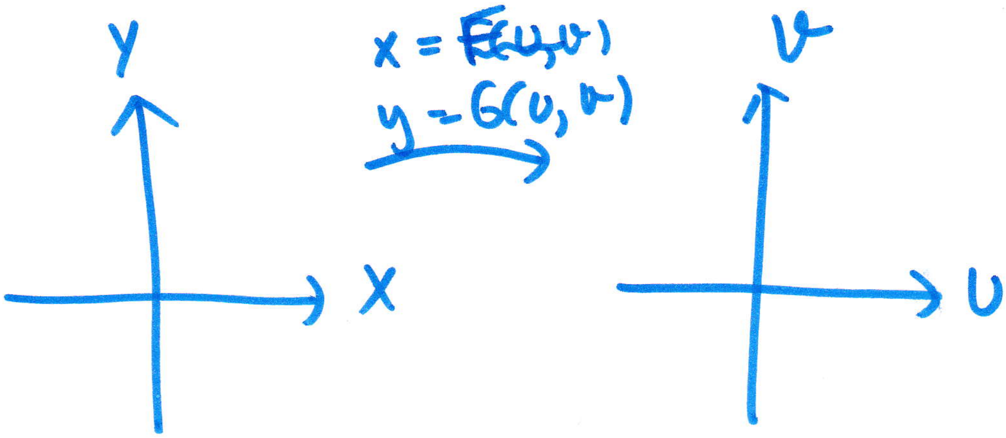
$$= f_{xx} F_r^2 + 2 f_{xy} F_r G_r + f_{yy} G_r^2 + f_x F_{rr} + f_y G_{rr} \quad \#$$

<u>PDEs</u> : Partial Differential Equations.

- $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}$ is said to be "Laplacian" of v., denoted by $\nabla^2 v$.
    - $\nabla^2 v = 0$ is s.t.b. "Laplacian Equation"
    - v satisfying $\nabla^2 v = 0$ is s.t.b "Harmonic f^{n}"
- $\nabla^4 f = 0$; f is s.t.b. "biharmonic function."

$$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} \leftarrow \text{operator}$$

$$\nabla f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j}$$



$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \longrightarrow ?$$

Ex 1.45
(อ่านเอง)

$$\boxed{\begin{aligned} x &= \rho\cos\theta \\ y &= \rho\sin\theta \end{aligned}}$$

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \Longrightarrow \frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial v}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 v}{\partial \theta^2} = 0.$$

~~Ex~~ Let $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$ be a PDEs s.t. $u \in C^2(\mathbb{R}^2)$

Suppose that we have transformation

$$x = s+t \quad , \quad y = s-t$$

Then $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0 \Longrightarrow ?$

แปลงได้ว่า $\boxed{\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial s \partial t}}$