$f(x,y) = \heartsuit + ✿ + \triangle + \cdots$

Thm (Taylor series of one variable)

Let $f(x)$ be differentiable up to order $n+1$ at every points in (a,b), and n-th order derivative is continuous in $[a,b]$. Let $x, x_0 \in [a,b]$ s.t $x \neq x_0$, Then $\exists\, \xi \in [x, x_0]$ s.t.

$$f(x) = f(x_0) + \frac{h}{1!} D_x f(x_0) + \frac{h^2}{2!} D_x^{(2)} f(x_0) + \dots + \frac{h^n}{n!} D_x^{(n)} f(x_n) + R_{n+1}$$

where $R_{n+1} = \dfrac{h^{n+1}}{(n+1)!} D_x^{n+1} F(\underbrace{x_0 + \theta h}_{\xi})$ for some $0<\theta<1$

R_{n+1} is said to be the remainder term.

Thm (Taylor series of two variables function).

Let $F(x,y)$ be differentiable up to order $n+1$ at every points (x,y) in the open region R which is determined that $F \in \mathcal{C}^n(\bar{R})$ in the closed region R. Then

$$F(x,y) = F(x_0,y_0) + \frac{(hD_x + kD_y)}{1!} F(x_0,y_0) + \frac{(hD_x + kD_y)^2}{2!} F(x_0,y_0) + \dots + \frac{(hD_x + kD_y)^n}{n!} F(x_0,y_0) + R_{n+1}$$

s.t. $R_{n+1} = \dfrac{(hD_x + kD_y)^{n+1}}{(n+1)!} F(x_0 + \theta h, y_0 + \theta k)$ for some $0 < \theta < 1$.

<u>Ex</u> Expand $x^2y+3y-2$ about $(x_0,y_0)=(1,-2)$

$$\boxed{\begin{aligned} h &= x-1 \\ k &= y+2 \end{aligned}}$$

$$F(x,y)=x^2y+3y-2 \quad ; \quad F(1,-2)=-10$$

$$F_x(x,y)=2xy \quad ; \quad F_x(1,-2)=-4$$

$$F_y(x,y)=x^2+3 \quad ; \quad F_y(1,-2)=4$$

$$F_{xx}(x,y)=2y \quad ; \quad F_{xx}(1,-2)=-4$$

$$F_{yy}(x,y)=0 \quad ; \quad F_{yy}(1,-2)=0$$

$$\left\{\begin{aligned} F_{xy}(x,y)&=2x \quad ; \quad F_{xy}(1,-2)=2 \\ F_{yx}(x,y)&=2x \quad ; \quad F_{yx}(1,-2)=2 \end{aligned}\right.$$

$$F_{xxx}(x,y)=0$$

$$F_{yyy}(x,y)=0$$

$$F_{xxy}(x,y)=2$$

$$F_{xyy}(x,y)=0$$

$$\begin{aligned} \therefore x^2y+3y-2 &= \cancel{-10+(2} \\ &= F(1,-2)+\left(hF_x(1,-2)+kF_y(1,-2)\right) \\ &\quad +\frac{1}{2}\left[h^2F_{xx}(1,-2)+2hkF_{xy}(1,-2)+k^2F_{yy}(1,-2)\right] \\ &\quad +\frac{1}{6}\left[h^3F_{xxx}(1,-2)+3h^2kF_{xxy}(1,-2)+\right. \\ &\qquad \left. 3hk^2F_{xyy}(1,-2)+k^3F_{yyy}(1,-2)\right] \\ &\quad +R_4 \end{aligned}$$

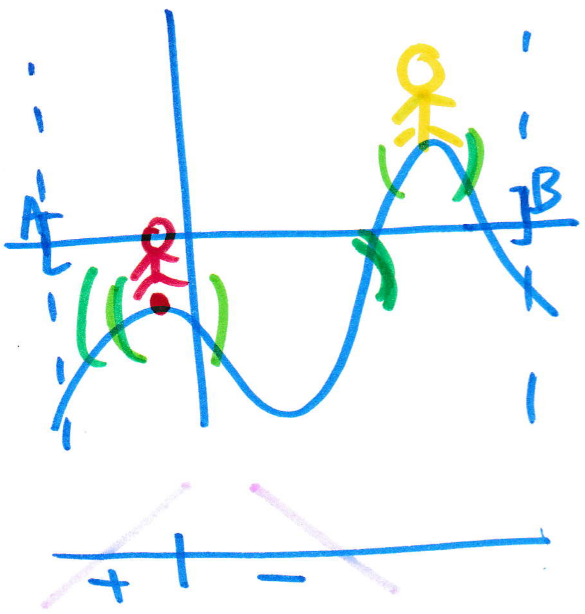
s.t. $R_4=\dfrac{(hD_x+kD_y)^4}{4!}F(1+\theta h,-2+\theta k)=0$

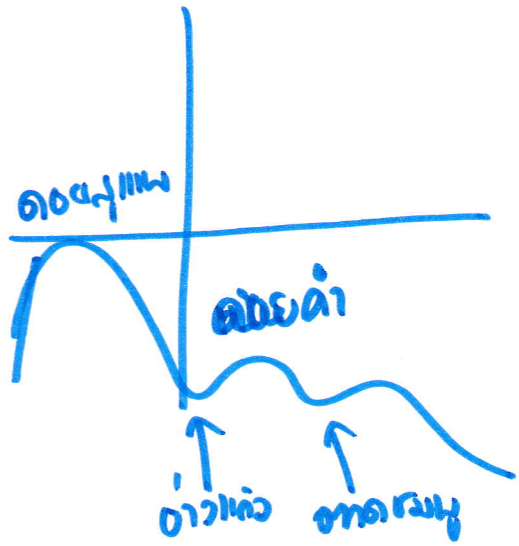
$$\begin{aligned} \therefore x^2y+3y-2 &= -10-4(x-1)+4(y+2)-2(x-1)^2+2(x-1)(y+2) \\ &\quad +(x-1)^2(y+2) \end{aligned} \quad \#$$

$f(x,y) = \cos x \cos y$ (กระจาย taylor รอบจุด $(0,0)$)

$$f\left(\frac{\pi}{1000}, \frac{\pi}{1000}\right) = \cos\left(\frac{\pi}{1000}\right)\cos\left(\frac{\pi}{1000}\right)$$

$\boxed{1.12}$ Maxima and Minima





<u>Definition</u> Let f be a function of two variables.

- f has an <u>absolute maximum value</u> at (x_0, y_0) in the region R of (xy-plane if

$$f(x_0, y_0) \geq f(x, y) \text{ for all } (x, y) \in R$$

- f has an <u>absolute minimum value</u> at (x_0, y_0) in the region R of xy-plane if

$$f(x_0, y_0) \leq f(x, y) \text{ for all } (x, y) \in R.$$

<u>Thm</u> (The extremum-value theorem of function of two variables)
Let R be a closed region in the xy-plane, and f be a function of two variables x, y which is cont. in R.
Then there exists a point in R s.t. f has an <u>absolute maximum</u> and a point in R s.t. f has an <u>absolute minimum</u>.

Definition Let f be a function of two variables.

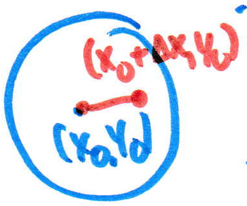
- f has a <u>relative maximum value</u> at (x_0, y_0) if there exists nbhd $N((x_0,y_0);r)$ s.t.
$$f(x,y) \le f(x_0,y_0) \quad \forall (x,y) \in N((x_0,y_0);r)$$
- f has a <u>relative minimum value</u> at (x_0,y_0) if there exists nbhd $N((x_0,y_0);r)$ s.t.
$$f(x,y) \geq f(x_0,y_0) \quad \forall (x,y) \in N((x_0,y_0);r)$$
- If f has a relative maximum or relative minimum, then f said to have a <u>relative extremum</u>

Theorem 1.23 If $f(x,y)$ exists for all $(x,y) \in N((x_0,y_0);r)$ and f has a local extrema at (x_0,y_0), and $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ exist, then
$$f_x(x_0,y_0) = f_y(x_0,y_0) = 0$$

Proof [relative maximum : (x_0,y_0)]

Suppose that f has relative maximum at (x_0, y_0).

$\therefore$ there exists $N((x_0,y_0);r)$ s.t. $f(x,y) \le f(x_0,y_0)$ $\forall (x,y) \in N((x_0,y_0);r)$



$\therefore$ For a point $(x_0+\Delta x, y_0) \in N((x_0,y_0);r)$,
$$f(x_0+\Delta x, y_0) \le f(x_0,y_0)$$
$$\therefore f(x_0+\Delta x, y_0) - f(x_0,y_0) \le 0$$

If $\Delta x > 0$; $\dfrac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x} \le 0$

$$\therefore \lim_{\Delta x \to 0^+} \frac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x} \le 0$$

If $\Delta x < 0$; $\dfrac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x} \geq 0$

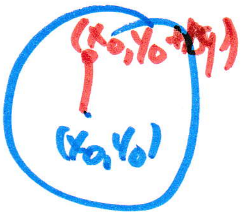
$\therefore \lim\limits_{\Delta x \to 0^-} \dfrac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x} \geq 0$

(ถ้าหาได้ : $\lim\limits_{\Delta x \to 0} \dfrac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x}$)

$\therefore \quad 0 \leq \lim\limits_{\Delta x \to 0^-} \dfrac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x} = \lim\limits_{\Delta x \to 0^+} \dfrac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x} \leq 0$

$\therefore \lim\limits_{\Delta x \to 0} \dfrac{f(x_0+\Delta x, y_0) - f(x_0,y_0)}{\Delta x} = 0$

Then $f_x(x_0,y_0) = 0$ #



<u>**Definition**</u> A point (x_0,y_0) at $f_x(x_0,y_0) = 0 = f_y(x_0,y_0)$ is s.t.b. a <u>critical point</u>.