Theorem 1.23 If $f(x,y)$ exists for all $(x,y)\in N((x_0,y_0);r)$ and f has a local extrema, and $f_x(x_0,y_0)$, $f_y(x_0,y_0)$ exist, then

$$f_x(x_0,y_0) = f_y(x_0,y_0) = 0$$

$p \Rightarrow q$

$\left(q \Rightarrow p \atop \text{converse} \right)$

บทกลับ ไม่จริง

Ex $f(x,y) = y^2 - x^2$ at $(0,0)$

$f_x(x,y) = -2x$ $\qquad f_y(x,y) = 2y$

$\therefore f_x(0,0) = 0$ $\qquad ; f_y(0,0) = 0$

$(0,0)$ is a saddle point of f.
(จุดอานม้า)

Theorem 1.24 let $f(x,y)$ be a function of x,y, the first and second derivatives are continuous in nbhd of $N((x_0,y_0);r)$, and assume that $f_x(x_0,y_0) = f_y(x_0,y_0) = 0$

Then ① If $\boxed{f_{xx}(x_0,y_0)\cdot f_{yy}(x_0,y_0) - f_{xy}^2(x_0,y_0)} > 0$ and $f_{xx}(x_0,y_0) > 0$, then $f(x_0,y_0)$ is a relative minimum of f.

② If $\boxed{f_{xx}(x_0,y_0) f_{yy}(x_0,y_0) - f_{xy}^2(x_0,y_0)} > 0$ and $f_{xx}(x_0,y_0) < 0$, then $f(x_0,y_0)$ is a relative maximum of f.

③ If $\boxed{f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-f_{xy}^2(x_0,y_0)<0}$, $\boxed{2}$
(x_0,y_0) is a saddle point.

④ If $\boxed{f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-f_{xy}^2(x_0,y_0)=0}$,
it cannot conclude!

[Discriminant : $f_{xx}f_{yy}-f_{xy}^2$]

Δ : $\boxed{f_{xx}f_{yy}-f_{xy}^2}$

- $\boxed{\Delta>0}$
  - $\boxed{f_{xx}>0}$ relative min.
  - $\boxed{f_{xx}<0}$ relative max
- $\boxed{\Delta<0}$ saddle point
- $\boxed{\Delta=0}$ cannot conclude!

Proof : Let $\phi(x_0,y_0)=f_{xx}(x_0,y_0)f_{yy}(x_0,y_0)-f_{xy}^2(x_0,y_0)$

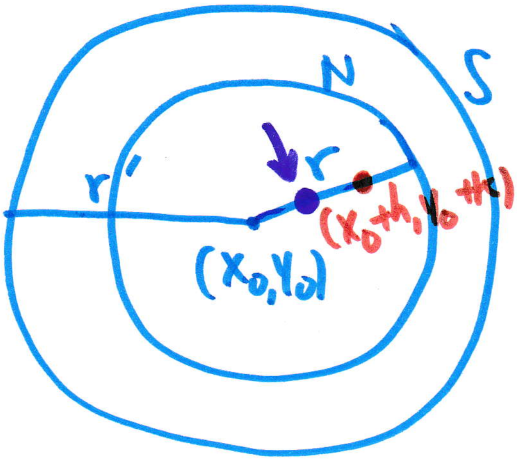
Assume that $f_x, f_{xy}, f_{yy}, f_{xy}$ are continuous in $N((x_0,y_0);r)$ for some $r\in\mathbb{R}$.

① Claim if $\phi(x_0,y_0)>0$ and $f_{xx}(x_0,y_0)>0$, then $f(x_0,y_0)$ is a relative minimum.

[$\exists N((x_0,y_0);r)$ ซึ่ง สำหรับจุดอื่นๆ (x^*,y^*) ใน $N((x_0,y_0);r)$, $f(x_0,y_0)\le f(x^*,y^*)$]

By the continuity of ϕ, if $\phi(x_0,y_0)>0$, then

$\exists N((x_0,y_0);r)$ s.t. $\phi(x_0,y_0)>0$ for all $(x,y)\in S((x_0,y_0);r)$



Let $h, k \neq 0$ s.t.

$(x_0+h, y_0+k) \in N((x_0,y_0); r) \subseteq S((x_0,y_0); r')$

Let (x,y) be a point on segment joining (x_0,y_0) and (x_0+h, y_0+k).

satisfying $\boxed{x = x_0 + ht}$ $\boxed{y = y_0 + kt}$ $0 \leq t \leq 1$

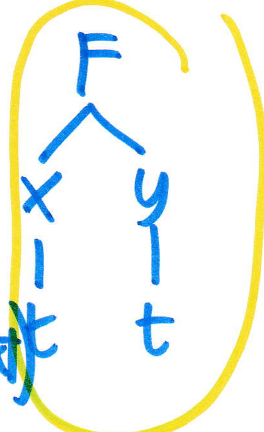
Let $F(t) = f(x_0 + ht, y_0 + kt)$

$F(0) = f(x_0, y_0)$, $\boxed{F(1) = f(x_0+h, y_0+k)}$

We expand the taylor series about $t = 0$

$$F(t) = F(0) + \frac{F'(0)}{1!}t + \frac{F''(\xi)}{2!}t^2 \ ; 0 \leq \xi \leq t$$

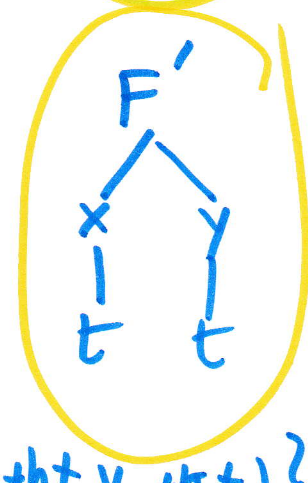
$$\therefore \boxed{F(1) = F(0) + \frac{F'(0)}{1!} + \frac{F''(\xi)}{2!}} \ ; 0 \leq \xi \leq 1$$



$$F'(t) = \frac{\partial F}{\partial x} \cdot \underset{h}{\boxed{\frac{\partial x}{\partial t}}} + \frac{\partial F}{\partial y} \underset{k}{\boxed{\frac{\partial y}{\partial t}}}$$

$$\therefore F'(t) = h f_x(x_0+ht, y_0+kt) + k f_y(x_0+ht, y_0+kt)$$

Thus, $\boxed{F'(0) = h f_x(x_0, y_0) + k f_y(x_0, y_0)}$



$$F''(t) = \frac{d}{dt} F'(t)$$

$$= \frac{\partial}{\partial x} F'(t) \cdot \overset{h}{\boxed{\frac{\partial x}{\partial t}}} + \frac{\partial}{\partial y} F'(t) \cdot \overset{k}{\boxed{\frac{\partial y}{\partial t}}}$$

$$= h \frac{\partial}{\partial x}\left\{h f_x(x_0+ht, y_0+kt) + k f_y(x_0+ht, y_0+kt)\right\}$$

$$+ k \frac{\partial}{\partial y}\left\{h f_x(x_0+ht, y_0+kt) + k f_y(x_0+ht, y_0+kt)\right\}$$

$$F''(t) = h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy} \quad \text{at } t.$$

at $(x_0+h\xi, y_0+k\xi)$,

$$F''(\xi) = h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy} \quad \text{at } (x_0+h\xi, y_0+k\xi)$$

Thus,

$$\boxed{\begin{aligned} f(x_0+h, y_0+k) = F(1) &= F(0) + \frac{F'(0)}{1!} + \frac{F''(\xi)}{2!} \\ &= f(x_0,y_0) + \left(h f_x(x_0,y_0) + k f_y(x_0,y_0)\right) \\ &\quad + \frac{1}{2!}\Big[h^2 f_{xx}(x_0+h\xi, y_0+k\xi) + \\ &\quad 2hk f_{xy}(x_0+h\xi, y_0+k\xi) + \\ &\quad k^2 f_{yy}(x_0+h\xi, y_0+k\xi)\Big] \end{aligned}}$$

$$\begin{aligned} \therefore\ \underline{f(x_0+h, y_0+k) - f(x_0,y_0)} &= \frac{1}{2}\left[h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right] \text{ at } (x_0+h\xi, y_0+k\xi) \\ &= \frac{1}{2} f_{xx}\left[h^2 + 2hk\frac{f_{xy}}{f_{xx}} + \frac{k^2}{f_{xx}} f_{yy}\right] \quad '' \\ &= \frac{1}{2} f_{xx}\left[\underline{\left(h^2 + 2h\left(\frac{k f_{xy}}{f_{xx}}\right) + \left(\frac{k f_{xy}}{f_{xx}}\right)^2\right.} - \left(\frac{k f_{xy}}{f_{xx}}\right)^2 \right. \\ &\qquad \left. + \frac{k^2}{f_{xx}} f_{yy}\right] \\ &= \frac{1}{2} f_{xx}\left[\left(h + \frac{k f_{xy}}{f_{xx}}\right)^2 - \left(\frac{k f_{xy}}{f_{xx}}\right)^2 + \frac{k^2}{f_{xx}} f_{yy}\right] \\ &\heartsuit\ \boxed{= \frac{1}{2} f_{xx}\left[\left(h + \frac{k f_{xy}}{f_{xx}}\right)^2 + k^2\left(\frac{f_{xx} f_{yy} - f_{xy}^2}{f_{xx}^2}\right)\right]} \end{aligned}$$

If $f_{xx} > 0$, and $f_{xx} f_{yy} - f_{xy}^2 > 0$ $\forall (x,y) \in N((x_0,y_0); r)$,

Then $\heartsuit \geq 0$ implies $f(x_0+h, y_0+k) \geq f(x_0,y_0)$

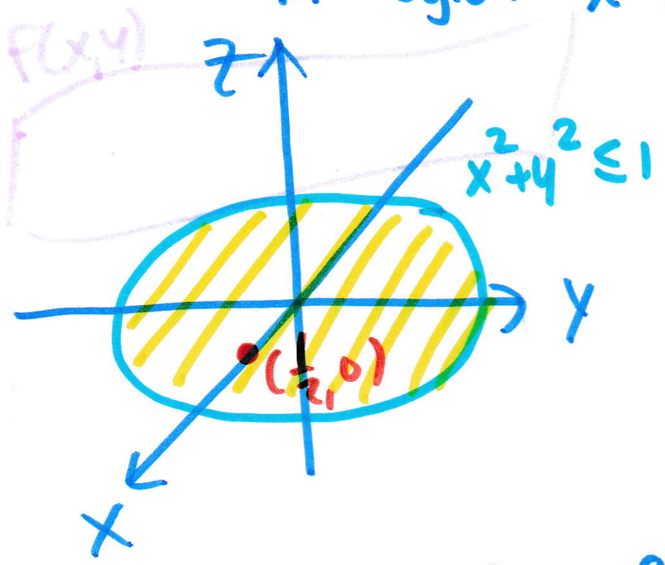
$\therefore f(x_0,y_0)$ is the local minimum of f. #

Ex $f(x,y) = x^2 + 2y^2 - x$

(1) Find the critical point(s)

(2) Verify the critical point(s)

(3) Find the absolute minimum or maximum in region $x^2+y^2 \le 1$



① Find (x_0,y_0) s.t.

$$f_x(x_0,y_0) = f_y(x_0,y_0) = 0$$

$$f_x(x,y) = 2x-1$$

$$f_y(x,y) = 4y$$

Set $f_x(x,y)=0$ | $f_y(x,y)=0$

$2x-1 = 0$ | $4y = 0$

$x = \frac{1}{2}$ | $y = 0$

$\therefore$ The critial point is $(\frac{1}{2},0)$

② $f_{xx}(x,y) = 2$

$f_{yy}(x,y) = 4$

$f_{xy}(x,y) = 0$

$$f_{xx}f_{yy} - f_{xy}^2 = (2)(4) - 0 = 8 > 0$$

$\therefore$ at $(\frac{1}{2},0)$ give the local minimum.

$$f(\tfrac{1}{2},0) = (\tfrac{1}{2})^2 - \tfrac{1}{2} = -\tfrac{1}{4}$$

#

3) We consider the boundary of the region.

$$x^2+y^2=1 \Rightarrow y=\pm\sqrt{1-x^2}$$

$$f(x,y)=x^2+2y^2-x$$

$$f(x,\pm\sqrt{1-x^2})=x^2+2(1-x^2)-x$$
$$=2-x-x^2$$

Let $g(x)=2-x-x^2$

$$g'(x)=-1-2x=0$$
$$x=-\frac{1}{2}$$

$g''(x)=-2<0 \Rightarrow$ at $x=-\frac{1}{2}$ gives the local maximum.

If $x=-\frac{1}{2}$, we have $y=\pm\sqrt{1-(-\frac{1}{2})^2}$

$$y=\pm\frac{\sqrt{3}}{2}$$

Thus, $(-\frac{1}{2},\frac{\sqrt{3}}{2})$, $(-\frac{1}{2},\frac{-\sqrt{3}}{2})$ gives the absolute maximum.

$$f(-\tfrac{1}{2},\pm\tfrac{\sqrt{3}}{2})=\frac{9}{4}$$

and at $(\frac{1}{2},0)$ gives the ~~local~~ absolute minimum, $-\frac{1}{4}$ #