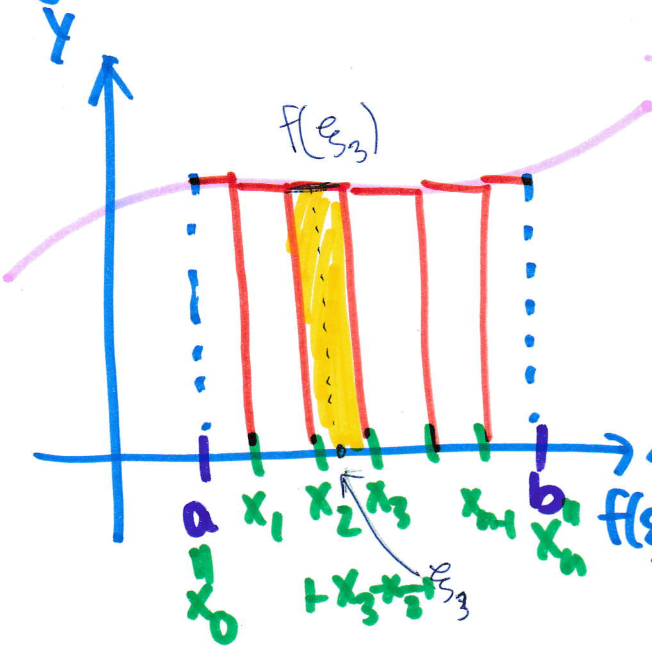
CHAPTER 2 Integrals

2.1 Definitions and Properties of definite integrals.

Let f be a function and continuous on $[a,b]$.
Suppose that $[a,b]$ is divided into n subintervals
by $a = x_0 < x_1 < \dots < x_n = b$



Let $\Delta_i x = x_i - x_{i-1}$
s.t. $i = 1, 2, \dots, n$
and $\xi_i \in [x_{i-1}, x_i]$

Then we find

$$f(\xi_1)(x_1 - x_0) + f(\xi_2)(x_2 - x_1) + \dots + f(\xi_n)(x_n - x_{n-1})$$
$$= \sum_{i=1}^{n} f(\xi_i)\Delta_i x$$

If we take $n \to \infty$, we represent.

$$\boxed{\lim_{n\to\infty} \sum_{i=1}^{n} f(\xi_i)\Delta_i x = \int_a^b f(x)\,dx} \quad \text{—— } (*)$$

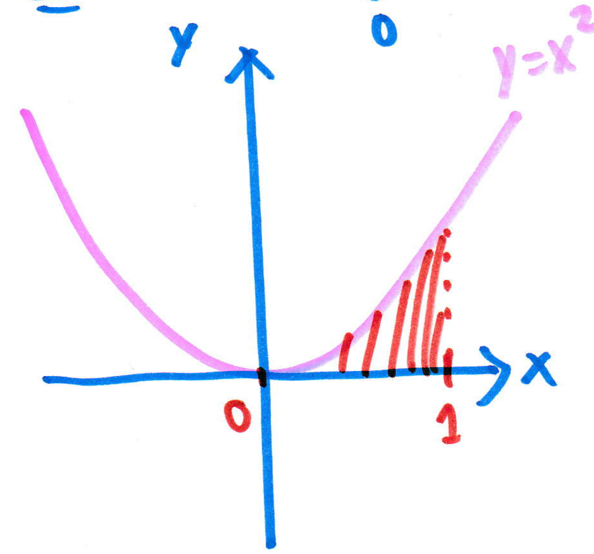
upper limit
integrand (ปริพัทธ์)
lower limit

$(*)$ is s.t.b. (said to be) the definite integral.
and the sum $\sum_{i=1}^{n} f(\xi_i)\Delta_i x$ is s.t.b. Riemann sum.

limit of $(*)$ exists iff $f(x)$ is <u>continuous</u> or <u>sectionally continuous</u>

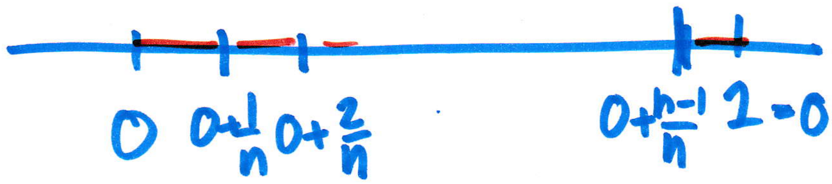
- If limit $(*)$ exists, then f is s.t.b. "<u>Riemann Integrable</u>"

Ex Find $\int_0^1 x^2\,dx$ by definition.



We divide $[0,1]$ to n subintervals s.t. the length of subintervals are equal.

$\therefore\ \Delta_i x_i = \frac{1}{n}$



Suppose that we choose ξ_i as the left-handed point of $[x_{i-1},x_i]$

Hence, $\sum_{i=1}^{n} f(\xi_i)\Delta_i x = f(\xi_1)\Delta_1 x + f(\xi_2)\Delta_2 x + \dots + f(\xi_n)\Delta_n x$

$$= f(0)\Delta_1 x + f(0+\tfrac{1}{n})\Delta_2 x + \dots + f(0+\tfrac{n-1}{n})\Delta_n x$$

$$= \sum_{i=1}^{n} f\left(0+\frac{(i-1)}{n}\right)\Delta_i x$$

$$= \sum_{i=1}^{n} f\left(0+\frac{(i-1)}{n}\right)\cdot\frac{1}{n}$$

$$= \sum_{i=1}^{n} \left(0+\frac{(i-1)}{n}\right)^2\cdot\frac{1}{n}$$

$$= \frac{1}{n^3}\sum_{i=1}^{n}(i-1)^2$$

$$\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} \qquad \sum_{i=1}^{n} i = \frac{n(n+1)}{2}}$$

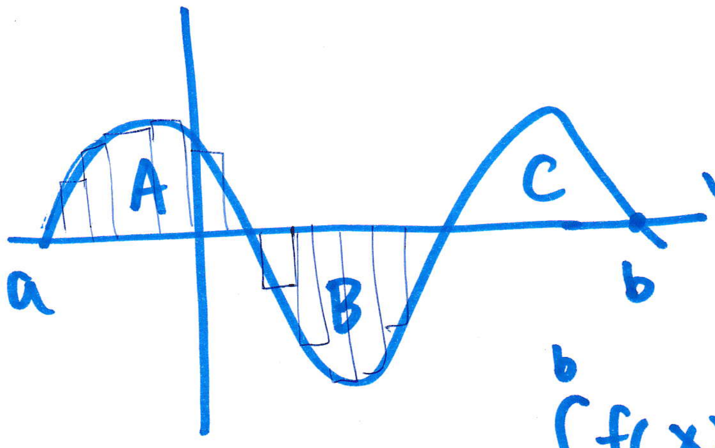
$$\lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\Delta_i x = \lim_{n\to\infty}\frac{1}{n^3}\sum_{i=1}^{n}(i^2-2i+1)$$

$$\frac{(n^2+n)(2n+1)}{6}$$

$$= \lim_{n\to\infty}\frac{1}{n^3}\left(\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n\right)$$

$$= \lim_{n\to\infty}\left(\frac{2n^3+3n^2+n}{6n^3} - 2\cancelto{0}{\frac{n^2+n}{2n^3}} + \cancelto{0}{\frac{n}{n^3}}\right) = \frac{1}{3}\ \#$$

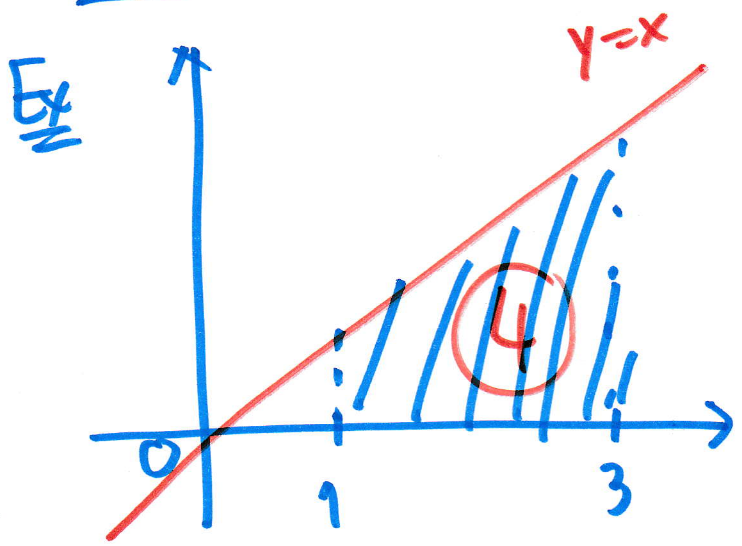
Meaning of $\int_a^b f(x)\,dx$?!?



$$\underbrace{\int_a^b f(x)\,dx}_{\text{Net signed area.}}$$

$$\int_a^b f(x)\,dx = (A+C) - B$$

Ex



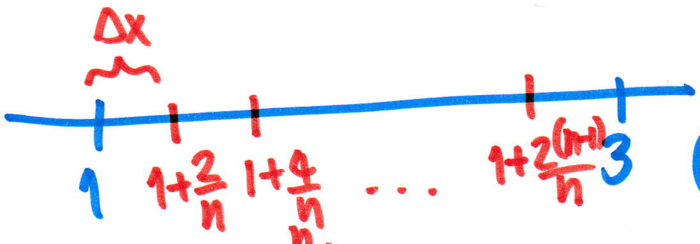
When approximating the following shaded area over $[1, 3]$ using Riemann sum with n equal subintervals.

① Identify $\Delta x = \frac{2}{n}$

② Show that the Riemann sum when using right endpoints approximation is $\sum_{k=1}^{n} \frac{2(n+2k)}{n^2}$

③ Given the shaded area in the figure is 4 unit squares. Use ② to find

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\cancel{2}(n+2k)}{n^2}. \quad \#$$



$$\therefore \text{Riemann Sum} = \sum_{k=1}^{n} f(\xi_i)\,\Delta_i x$$

$$= \sum_{k=1}^{n} \left(1 + \frac{2k}{n}\right)\cdot\frac{2}{n}$$

$$= \sum_{k=1}^{n} \left(\frac{2(n+2k)}{n^2}\right)$$

$$\text{Area} = 4 = \lim_{n\to\infty}\left(\sum_{k=1}^{n} \frac{2(n+2k)}{n^2}\right) \Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} \frac{(n+2k)}{n^2} = 2 \quad \#$$

Theorem 2.1 If $f(x)$ and $g(x)$ are integrable in $[a,b]$, then

ปลวก

① $\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$

② $\int_a^b k f(x)\,dx = k \int_a^b f(x)\,dx$

③ $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ s.t. f is integrable on $[a,c]$ and $[c,b]$ and $a \le c \le b$.

④ $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

⑤ $\int_a^a f(x)\,dx = 0$

⑥ If $x \in [a,b]$, $m \le f(x) \le M$, then
$$m(b-a) \le \int_a^b f(x)\,dx \le M(b-a)$$

⑦ If $f(x) \geq g(x)$ $\forall x \in [a,b]$, then $\cancel{\int f(x)dx}$
$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$$

⑧ $\left| \int_a^b f(x)\,dx \right| \le \int_a^b |f(x)|\,dx$

Proof ⑦ Assume that $f(x) \geq g(x)$ for all $x \in [a,b]$.

Let $a = x_0 < x_1 < \ldots < x_n = b$ be points s.t. $x_0, \ldots, x_n$ divide $[a,b]$ into n subintervals.

Denote $\Delta_i x = x_i - x_{i-1}$

For each i, let $\xi_i \in [x_i, x_{i-1}]$.

Then $f(\xi_i) \geq g(\xi_i)$

$\therefore \quad f(\xi_i)\Delta_i x \geq g(\xi_i)\Delta_i x.$

Hence, $\displaystyle\sum_{i=1}^{n} f(\xi_i)\Delta_i x \geq \sum_{i=1}^{n} g(\xi_i)\Delta_i x$

Then $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\Delta_i x \geq \lim_{n\to\infty}\sum_{i=1}^{n} g(\xi_i)\Delta_i x$

Thus, $\displaystyle\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$ #

ภาคขยายทั่วไป. (generalized)

⑧ Since $\displaystyle\left|\sum_{i=1}^{n} f(\xi_i)\Delta_i x\right| \leq \sum_{i=1}^{n} |f(\xi_i)\Delta_i x|$

$\displaystyle\lim_{n\to\infty}\left|\sum_{i=1}^{n} f(\xi_i)\Delta_i x\right| \leq \lim_{n\to\infty}\sum_{i=1}^{n} |f(\xi_i)\Delta_i x|$

$\displaystyle\therefore \left|\lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\Delta_i x\right| \leq \lim_{n\to\infty}\sum_{i=1}^{n} |f(\xi_i)\Delta_i x|$

$\displaystyle\therefore \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$ #

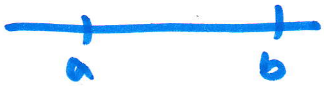
Ex $\lim\limits_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f\left(\frac{i}{n}\right) = \int_0^1 f(x)\,dx$

$\frac{1}{n} \to \frac{b-a}{n}$ $\quad 0+\frac{i(b-a)}{n} \Rightarrow a=0,\ b=1$

Ex Let $f(x)$ be a continuous on $[a,b]$. Prove that

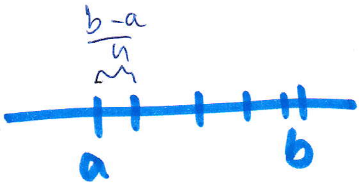
$$\lim_{n\to\infty} \frac{(b-a)}{n}\sum_{i=1}^{n} f\left(a+\frac{i(b-a)}{n}\right) = \int_a^b f(x)\,dx.$$

Proof Assume that f is cont. on $[a,b]$.



We divide $[a,b]$ into n equal subintervals.

$\Delta_i x = \frac{b-a}{n}.$



Choose $\xi_i \in [x_{i-1}, x_i]$ where

$\xi_i = a + \frac{i(b-a)}{n}$ s.t. $i=1,\ldots n$

(เลือกจุดปลายช่วงด้านขวา)

$$\therefore \int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\Delta_i x = \lim_{n\to\infty}\frac{b-a}{n}\sum_{i=1}^{n} f\left(a+\frac{i(b-a)}{n}\right)$$

$\#$

Ex $\lim\limits_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}\right) = \lim\limits_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i}$

$$= \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n(1+i/n)}$$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+i/n}$$

$\Delta x = \frac{1}{n}$ $\quad \frac{b-a}{n} = \frac{1}{n} \Rightarrow a=0,\ b=1$

$$= \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+i/n}$$

$f(x) = \frac{1}{1+x}$

$$= \int_0^1 \frac{1}{1+x}\,dx. \quad \#$$

Ex HW

$$\lim_{n\to\infty} \frac{1}{n}\left(\sin\frac{t}{n} + \sin\frac{2t}{n} + \dots + \sin\frac{nt}{n}\right) = \boxed{}$$

Ex Prove that $\displaystyle\int_0^{\pi/2} \sqrt{x\sin x}\,dx \leq \frac{2}{3}\left(\frac{\pi}{3}\right)^{3/2}$

$$\int_0^{\pi/2} \sqrt{x\sin x}\,dx = \left|\int_0^{\pi/2} \sqrt{x\sin x}\,dx\right|$$

$$\overset{8}{\leq} \int_0^{\pi/2} \left|\sqrt{x\sin x}\right| dx$$

$$\boxed{\left|\sqrt{\sin x}\right| \leq 1} \ ♥$$

$$\overset{♥}{\leq} \int_0^{\pi/2} (\sqrt{x})\,dx$$

$$= \left.\frac{2x^{3/2}}{3}\right|_0^{\pi/2}$$

$$= \frac{2}{3}\left(\frac{\pi}{2}\right)^{3/2}$$

#

HW Show that $\displaystyle\left|\int_1^{\sqrt{3}} \frac{e^{-x}\sin x}{x^2+1}\,dx\right| \leq \frac{\pi}{12e}$

Hint: ดูขอบเขตของ e^{-x} จาก $x = 1$ ถึง $\sqrt{3}$ ว่าเป็นอย่างไร เพื่อให้สอดคล้องกับ $|e^{-x}|$ . . .