Ex $\lim\limits_{n\to\infty} \frac{1}{n}\left(\sin\frac{t}{n} + \sin\frac{2t}{n} + \dots + \sin\frac{nt}{n}\right)$ #

$\Delta_i x = \frac{t-0}{n}$

$a=0, b=t$

4*

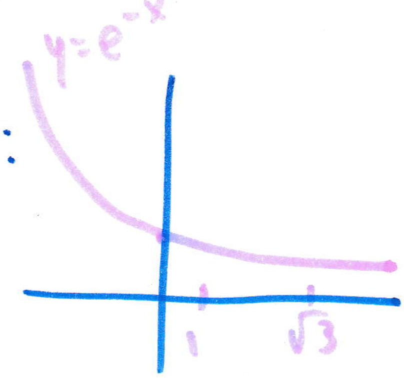
$= \lim\limits_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} \sin\left(\frac{it}{n}\right)$

$= \frac{1}{t}\lim\limits_{n\to\infty} \frac{t}{n}\sum_{i=1}^{n} \sin\left(\frac{it}{n}\right)$

$= \frac{1}{t}\int_0^t \sin x \, dx$ #

Ex $\left|\int_1^{\sqrt3} \frac{e^{-x}\sin x}{x^2+1}dx\right| \le \frac{\pi}{12e}.$

Hint:



$$\left|\int_1^{\sqrt3} \frac{e^{-x}\sin x}{x^2+1}dx\right| \le \int_1^{\sqrt3} \left|\frac{e^{-x}\sin x}{x^2+1}\right|dx$$

$$\le \frac{1}{e}\int_1^{\sqrt3} \frac{1}{x^2+1}dx$$

$$= \frac{1}{e}\left[\arctan x\right]_1^{\sqrt3}$$

$$= \frac{1}{e}\left[\frac{\pi}{3} - \frac{\pi}{4}\right]$$

$$= \frac{\pi}{12e}$$ #

ดู bound ของ $|e^{-x}|$
ในช่วง $[1,\sqrt3]$

$e^{-\sqrt3} \le e^{-x} < e^{-1}$

$-\frac{1}{e} \le \frac{1}{e^{\sqrt3}} \le e^{-x} \le e^{1}$

$\Downarrow$

$|e^{-x}| \le \frac{1}{e}$

(2.3) Fundamental Theorem of Calculus.

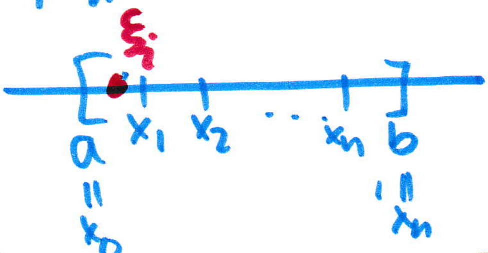
Theorem 2.6 Fundamental Thm of Calculus : First form.

If $f(x)$ is continuous on $[a,b]$ and $F(x)$ is a function s.t. $F'(x) = f(x)$, then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Proof Divide $[a,b]$ with $a = x_0, x_1, \dots, x_n$ s.t.

$a = x_0 < x_1 < \dots < x_n = b$



and let $\xi_i \in [x_{i-1}, x_i]$;

$\Delta_i x = x_i - x_{i-1}$

Mean value theorem for derivative:

Thm If f is cont on $[a,b]$, differentiable on (a,b)

Then $\exists c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

By MVT(derivation), for each subinterval $[x_{i-1}, x_i]$,

$\exists \xi_i \in [x_i, x_{i-1}]$, $\boxed{F(x_i) - F(x_{i-1}) = F'(\xi_i)(x_i - x_{i-1})}$

$F(x_i) - F(x_{i-1}) = f(\xi_i)\Delta_i x$

Hence:

$$\left.\begin{aligned} F(x_1) - F(x_0) &= f(\xi_1)\Delta_1 x \\ F(x_2) - F(x_1) &= f(\xi_2)\Delta_2 x \\ &\vdots \\ F(x_n) - F(x_{n-1}) &= f(\xi_n)\Delta_n x \end{aligned}\right\}$$

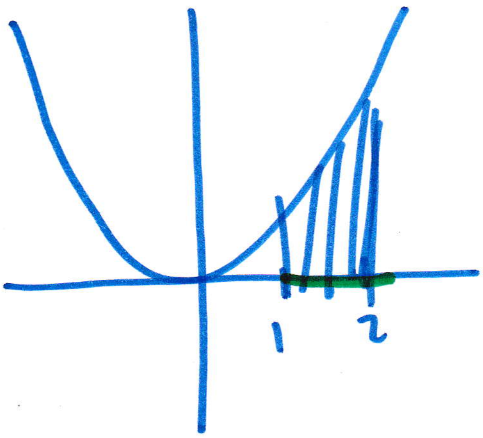
Hence, $F(x_n) - F(x_0) = \sum_{i=1}^{n} f(\xi_i)\Delta_i x$

Take limit $n \to \infty$

$$F(b)-F(a) = \lim_{n\to\infty}\left[F(b)-F(a)\right] = \lim_{n\to\infty}\sum_{i=1}^{n} f(\xi_i)\Delta_i x = \int_a^b f(x)\,dx. \ \#$$

Ex $\displaystyle\int_1^2 x^2\,dx =$

$\left(\text{Antiderivative of } x^2 \text{ is } \dfrac{x^3}{3}+C \qquad F(x) = \dfrac{x^3}{3}+C\right)$



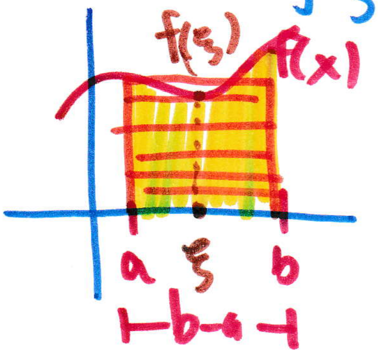
$$\int_1^2 x^2\,dx = F(2)-F(1)$$

$$= \left(\frac{8}{3}+\cancel{C}\right)-\left(\left(\frac{1}{3}\right)+\cancel{C}\right) = \frac{7}{3} \ \#$$

Theorem 2.3 First mean value theorem for integral

If $f(x)$ is continuous on $[a,b]$, then

$\exists\, \xi \in (a,b)$ s.t. $\boxed{\displaystyle\int_a^b f(x)\,dx =} (b-a)f(\xi)$.

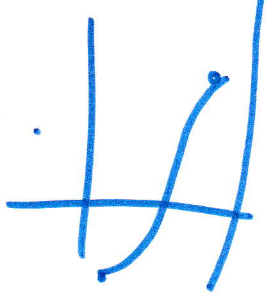


(ของที่ต้องใช้ ① extreme value theorem

$\Rightarrow$ If f is cont on $[a,b]$, then f has both minimum and maximum on $[a,b]$.

② Intermediate value theorem

$\Rightarrow$ If f is cont on $[a,b]$, k is a number s.t. $f(a) \le k \le f(b)$, then $\exists\, x \in [a,b]$ s.t. $f(x) = k$.)



Proof: Suppose that f is cont on $[a,b]$
By EV.T, we have, $\exists m, M$ s.t.

$$m \le f(x) \le M$$

Since $$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

$$m \le \frac{\int_a^b f(x)dx}{b-a} \le M$$

Since f is cont. on $[a,b]$, by IVT, we have
$\exists \xi \in [a,b]$ s.t. $f(\xi) = \dfrac{\int_a^b f(x)dx}{b-a}$

$$\therefore \int_a^b f(x)dx = f(\xi)(b-a)$$ #

Theorem 2.6 (Fundamental Thm of Calculus : Second form)
let $f(x)$ be integrable in $[a,b]$ and $F(x) = \int_a^x f(t)dt$,
and f be continuous in $[a,b]$. Then $F'(x) = f(x)$., i.e.

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Proof
$$\frac{F(x+h)-F(x)}{h} = \frac{1}{h}\left[\int_a^{x+h} f(t)dt - \int_a^x f(t)dt\right]$$

$$= \frac{1}{h}\left[\int_x^{x+h} f(t)dt\right]$$

MVT (Int) $\exists \xi \in (x, x+h)$

$$= \frac{1}{\cancel{h}}\left[f(\xi)[\cancel{(x+h)-x}]\right] = f(\xi)$$

$$F'(x) = \lim_{h\to 0}\frac{F(x+h)-F(x)}{h} = \lim_{h\to 0} f(\xi) = f(x)$$ #

Ex $\frac{d}{dx}\int_a^x e^{t^2}\sin t\,dt = e^{x^2}\sin x$

⊛ $\int f(x)dx = \int_a^x f(t)dt + C$

$\searrow \int_a^x f(u)du$

$$\frac{d}{dx}\int_{u(x)}^{v(x)} f(t)dt = f(t)\frac{dt}{dx}\Big|_{t=u(x)}^{v(x)} = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$$

Ex $\frac{d}{dx}\int_x^{x^2} f(t)dt$ เมื่อ $f(t) = \frac{\sin t}{t}$

$$\Rightarrow \frac{d}{dx}\int_x^{x^2}\frac{\sin t}{t}dt = \frac{\sin x^2}{x^2}\frac{d(x^2)}{dx} - \frac{\sin x}{x}\frac{dx}{dx}$$

$$= \frac{2\sin x^2}{x} - \frac{\sin x}{x} \quad \#$$

Integration by substitution

$\boxed{x = g(t)}$; $\int f(x)dx = \int f(g(t))g'(t)dt$

$\Downarrow$

$$\int_a^b f(x)dx = \int_\alpha^\beta f(g(t))g'(t)dt$$

suppose that $t = \alpha \Rightarrow a = g(\alpha) \Rightarrow \alpha = g^{-1}(a)$

$t = \beta \Rightarrow b = g(\beta) \Rightarrow \beta = g^{-1}(b)$

Ex $\displaystyle\int_{e}^{e^2} \frac{dx}{x(\ln x)^3} \longrightarrow \int_{\bullet}^{\blacktriangle} f(u)\,du$

$u = \ln x$ <u>ขอบเขต</u> $x = e \Rightarrow u = \ln e = 1$

$du = \frac{1}{x}dx$ $x = e^2 \Rightarrow u = \ln e^2 = 2$

$$\int_{e}^{e^2} \frac{dx}{x(\ln x^3)} = \int_{1}^{2} \frac{du}{u^3} = \left[-\frac{1}{2u^2}\right]_1^2 = \frac{3}{8} \quad \#$$

Ex Find an appropriate choice u, a, b, f for integration by substitution

$$\int_{0}^{\sqrt{12}} 6x\sqrt{x^2+4}\,dx = \int_{a}^{b} f(u)\,du$$

$a =$ 4 $u =$ x^2+4

$b =$ 16 $f =$ $3\sqrt{u}$

$u = x^2+4$

$du = 2x\,dx$ $\Rightarrow \int 6x\sqrt{u}\,dx$ $3du$ $= \int 3\sqrt{u}\,du$

$\hookrightarrow x = 0 \Rightarrow u = 4$

$x = \sqrt{12} \Rightarrow u = 16$