Mean Value theorem for integral (MVT)

Thm If f is cont on $[a,b]$, then $\exists \xi \in (a,b)$ s.t.

$$\int_a^b f(x)dx = (b-a)f(\xi)$$

Intermediate-value theorem (IVT)

Thm If f is cont. $[a,b]$, k is a number s.t. $f(a) \le k \le f(b)$, then $\exists x \in [a,b]$ s.t. $f(x) = k$.

Extreme Value theorem (EVT)

Thm If f is cont. on $[a,b]$, then f has both minimum and maximum on $[a,b]$.

Thm 2.2 (Generalized first mean value theorem for integral)
Let $f(x)$ and $g(x)$ be continuous functions on $[a,b]$, $g(x) \neq 0$, and sign of $g(x)$ does not change in $[a,b]$, then $\exists \xi \in (a,b)$ s.t.

$$\int_a^b f(x)g(x)dx = f(\xi)\int_a^b g(x)dx.$$

Proof: (Case $g(x) > 0$) Suppose that $g(x) > 0$ in $[a,b]$.
Since $f(x)$ is cont on $[a,b]$, (EVT), $\exists m, M$ s.t.

absolute minimum $\rightarrow$ $\boxed{m \le f(x) \le \boxed{M}}$ $\leftarrow$ absolute maximum.

Since $g(x) > 0$,

$$\therefore \quad mg(x) \le f(x)g(x) \le Mg(x)$$

$$\therefore \quad m\int_a^b g(x)dx \le \int_a^b f(x)g(x)dx \le M\int_a^b g(x)dx$$

$$\therefore\ m \le \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \le M$$

(f cont)

By IVT., $\exists\ \xi \in [a,b]$ s.t.

$$f(\xi) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}. \quad \text{—}(*)$$

We consider that we can choose $\xi \in (a,b)$ satisfying $(*)$

<u>Case 1</u>: $f(a) \ne \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \ne f(b) \Rightarrow \xi \ne a, \xi \ne b.$

<u>Case 2</u> $f(a) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx} \Rightarrow \int_a^b f(x)g(x)dx = f(a)\int_a^b g(x)dx$

Hence $\int_a^b [f(x)-f(a)]g(x)\,dx = 0$

Since $g(x)>0$, $f(x)-f(a) \geq 0$, $f(x), g(x)$ cont $[a,b]$,

$\therefore\ f(x) = f(a) \quad \forall x \in [a,b]$.

$\therefore \exists\ \xi \in (a,b)$ Satisfying $(*)$

<u>Case</u> $f(b) = \frac{\int_a^b f(x)g(x)dx}{\int_a^b g(x)dx}$, similarly to case 2. #

Ex $\displaystyle\int_0^{1/2} \frac{1}{1-x^2}\,dx$: Find the upper bound, lower bound

ห้ามทำ $\displaystyle\frac{1}{1-x^2} = \frac{A}{1-x} + \frac{B}{1+x}$

$$\left.\begin{aligned} 1 &= A(1+x) + B(1-x) \\ 1 &= A+B + (A-B)x \end{aligned}\right\} \quad \begin{aligned} A+B &= 1 \Rightarrow A=\tfrac{1}{2},\ B=\tfrac{1}{2} \\ A-B &= 0 \Rightarrow A=B \end{aligned}$$

$$\int \frac{1}{1-x^2}\,dx = \frac{1}{2}\int\left(\frac{1}{1-x}\right)dx + \frac{1}{2}\int\left(\frac{1}{1+x}\right)dx$$

$$= \frac{1}{2}\left[\ln|1-x| + \ln|1+x|\right]$$

$$\therefore \int_0^{1/2} \frac{1}{1-x^2}\,dx = \frac{1}{2}\left[\ln|1-x| + \ln|1+x|\right]_0^{1/2} \ldots$$

Sol^{n} Consider whether $f(x) = \dfrac{1}{1-x^2}$ is cont $[0, \frac{1}{2}]$ or not.

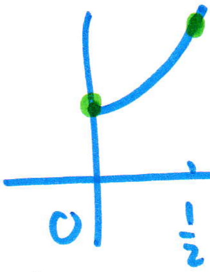
Ans: YES!

By MVT, $\exists\, \xi \in (0, \frac{1}{2})$ s.t.

$$\int_0^{1/2} \frac{1}{1-x^2}\,dx = \left(\frac{1}{2} - 0\right) f(\xi) = \frac{1}{2} f(\xi)$$

Since $\xi \in (0, \frac{1}{2})$, $f(x)$ ~~function~~ is increasing function, on $[0,\frac{1}{2}]$ we have

$$\therefore\ 1 \le f(\xi) \le \frac{4}{3}$$



Hence, $\displaystyle \frac{1}{2} \le \frac{1}{2} f(\xi) \le \frac{2}{3}$

$$\Downarrow$$

$$\frac{1}{2} \le \int_0^{1/2} \frac{1}{1-x^2}\,dx \le \frac{2}{3}$$

#

ทด $f(x) = \dfrac{1}{1-x^2}$ บน $[0, \frac{1}{2}]$

$$f'(x) = \frac{d}{dx}(1-x^2)^{-1}$$

$$= -1(1-x^2)^{-2}(-2x)$$

$$= \frac{2x}{(1-x^2)^2}$$

$\therefore$ บน $[0, \frac{1}{2}]$, จุดวิกฤตคือ $x = 0$

$\underset{0}{|}\ +\ +\ +$

Theorem If f is cont on $[a,b]$, then $\exists \xi \in (a,b)$ s.t.

$$\int_a^b f(x)\,dx = (b-a)\,f(\xi)$$

Proof Set $g(x) = 1$ #

Ex 2.7. Find upper bound and lower bound of the integral

$$\int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \quad ; \text{ for all } k^2 < 2$$

Sol^{n} By MVT (G1); choose $f(x) = \frac{1}{\sqrt{1-k^2x^2}}$, $g(x) = \frac{1}{\sqrt{1-x^2}}$

Remark that ~~Here,~~ the sign of $g(x)$ does not change in $[0, \frac{1}{2}]$.

Thus, $\exists \xi \in (0, \frac{1}{2})$ s.t.

$$\int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = \frac{1}{\sqrt{1-k^2\xi^2}} \int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx$$

$$= \frac{1}{\sqrt{1-k^2\xi^2}} \left[\arcsin x\right]_0^{1/2}$$

$$= \frac{1}{\sqrt{1-k^2\xi^2}} \left[\arcsin \tfrac{1}{2} - \arcsin 0\right]$$

$$= \frac{1}{\sqrt{1-k^2\xi^2}} \left[\frac{\pi}{6}\right] = \frac{\pi}{6\sqrt{1-k^2\xi^2}}$$

$$= \frac{\pi}{6} f(\xi)$$

CHECK : $f(\xi)$ is increasing f^{n} / decreasing f^{n} on $[0, \frac{1}{2}]$

⊛ f is increasing f^{n} on $[0, \frac{1}{2}]$ $\forall k$, $k^2 < 2 \Rightarrow 1 \leq f(\xi) \leq \frac{1}{\sqrt{1-\frac{k^2}{4}}}$

⊛ $f(0) = 1$, $f(\frac{1}{2}) = \frac{1}{\sqrt{1-k^2/4}}$

$$\therefore \quad \frac{\pi}{6} \le \frac{\pi}{6} f(\xi) \le \frac{\pi}{6} \cdot \frac{1}{\sqrt{1-\frac{k^2}{4}}}$$

$$\|$$

$$\frac{\pi}{6} \le \int_0^{1/2} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \le \frac{\pi}{6} \cdot \frac{1}{\sqrt{1-\frac{k^2}{4}}} \quad \#$$

Thm 2.4 (Generalized 2nd MVT for integral).

If $f(x)$, $g(x)$ and $g'(x)$ are cont. on $[a,b]$ and $g(x)$ is nonincreasing f^n or nondecreasing f^n (i.e. the sign of $g'(x)$ does not change), then

$\exists\, \xi \in (a,b)$ s.t.

$$\int_a^b f(x)g(x)\,dx = g(a)\int_a^{\xi} f(x)\,dx + g(b)\int_{\xi}^{b} f(x)\,dx.$$

Proof: Let $F(x) = \int_a^x f(t)\,dt$. $\boxed{\frac{dF(x)}{dx} = f(x) \Rightarrow dF(x) = f(x)\,dx \quad \text{FTC2}}$

$$\therefore \int_a^b f(x)g(x)\,dx = \int_a^b g(x)\,dF(x)$$

choose $u = g(x)$; $dv = f(x)\,dx = dF(x)$

$\therefore du = g'(x)\,dx$; $v = F(x)$

$$= [g(x)F(x)]_{x=a}^{b} - \int_a^b F(x)g'(x)\,dx$$

$$= [g(b)F(b) - g(a)F(a)] - \int_a^b F(x)g'(x)\,dx$$

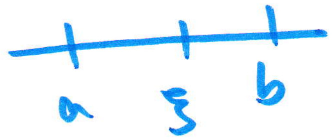
$$\overset{\text{MVT(GI)}}{=} [g(b)F(b) - g(a)\underline{F(a)}] - F(\xi)\int_a^b g'(x)\,dx$$

for some $\xi \in (a,b)$ $\otimes$

Since $F(a) = \int_a^a f(t)\,dt = 0$, we have

$$(*) \quad = g(b)F(b) - F(\xi)[g(b)-g(a)]$$

$$= g(b)\int_a^b f(t)dt - g(b)\int_a^{\xi} f(t)dt + g(a)\int_a^{\xi} f(t)dt$$



$$= g(a)\int_a^{\xi} f(t)dt + g(b)\left[\int_a^b f(t)dt - \int_a^{\xi} f(t)dt\right]$$

$$= g(a)\int_a^{\xi} f(t)dt + g(b)\int_{\xi}^{b} f(t)dt$$

$$= g(a)\int_a^{\xi} f(x)dx + g(b)\int_{\xi}^{b} f(x)dx \quad \#$$

Thm 2.5 (Bonnet's second MVT for integral)

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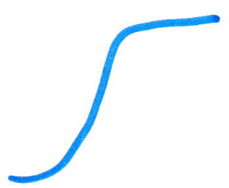
If $f(x)$ and $g(x)$ are cont. on $[a,b]$, and if $g(x) \geq 0$ is nonincreasing function, and not a constant function, then $\exists\, \xi \in (a,b)$ s.t.

$$\int_a^b f(x)g(x)dx = g(a)\int_a^{\xi} f(x)dx$$

and if $g(x) \geq 0$ is nondecreasing function and not a constant function, then $\exists\, \xi \in (a,b)$ s.t.

$$\int_a^b f(x)g(x)dx = g(b)\int_{\xi}^{b} f(x)dx$$



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