Ex $$\frac{d}{dx}\int_2^x \frac{\sin(e^{t^2})}{t}\,dt = \frac{\sin(e^{x^2})}{x}$$

Ex Let $f(x,t) = (2x+t^3)^2$

Consider $\int_0^1 f(x,t)\,dx = \int_0^1 (2x+t^3)^2\,dx$

Find $\frac{d}{dt}\int_0^1 f(x,t)\,dt = \frac{d}{dt}\int_0^1 (2x+t^3)^2\,d\cancel{x}$

$$= \frac{d}{dt}\left[\frac{(2x+t^3)^2}{6}\Big|_{x=0}^{1}\right]$$

$$= \frac{d}{dt}\left[\frac{4}{3}+2t^3+t^6\right]$$

$$= 6t^2+6t^5 \quad \#$$

<u>Consider</u> $\int_0^1 \frac{\partial}{\partial t} f(x,t)\,dx = \int_0^1 \frac{\partial}{\partial t}(2x+t^3)^2\,dx$

$$= \int_0^1 2(2x+t^3)(3t^2)\,dx$$

$$= \int_0^1 (12t^2x+6t^5)\,dx$$

$$= 6t^2x^2+6t^5x\Big|_{x=0}^{1}$$

$$= 6t^2+6t^5 \quad \#$$

Thm 2.7 Leibnitz's Theorem

Let $\phi(\alpha) = \int_{u(\alpha)}^{v(\alpha)} f(t,\alpha)\,dt$ s.t. $u \le \alpha \le v$.

If $f(t,\alpha)$ and $\frac{\partial f}{\partial \alpha}$ are continuous functions in the plane $t\alpha$ s.t. $u \le t \le v$, $a \le \alpha \le b$.

If u, v are continuous function and the derivatives are continuous for all $a \le \alpha \le b$, then

$$\frac{d}{d\alpha}\phi(\alpha) = \int_{u(\alpha)}^{v(\alpha)} \frac{\partial f}{\partial \alpha}\,dt + f(v,\alpha)\frac{dv}{d\alpha} - f(u,\alpha)\frac{du}{d\alpha}.$$

Proof Consider

$$\Delta\phi(\alpha) = \phi(\alpha+\Delta\alpha) - \phi(\alpha)$$

$$= \int_{u(\alpha+\Delta\alpha)}^{v(\alpha+\Delta\alpha)} f(t,\alpha+\Delta\alpha)\,dt - \int_{u(\alpha)}^{v(\alpha)} f(t,\alpha)\,dt \quad \text{—} \circledast$$

$$= \left(\int_{u(\alpha+\Delta\alpha)}^{u(\alpha)} f(t,\alpha+\Delta\alpha)\,dt + \int_{u(\alpha)}^{v(\alpha)} f(t,\alpha+\Delta\alpha)\,dt + \int_{v(\alpha)}^{v(\alpha+\Delta\alpha)} f(t,\alpha+\Delta\alpha)\,dt\right)$$

$$- \int_{u(\alpha)}^{v(\alpha)} f(t,\alpha)\,dt$$

$$= \int_{u(\alpha)}^{v(\alpha)} [f(t,\alpha+\Delta\alpha) - f(t,\alpha)]\,dt - \int_{u(\alpha)}^{u(\alpha+\Delta\alpha)} f(t,\alpha+\Delta\alpha)\,dt$$

$$+ \int_{v(\alpha)}^{v(\alpha+\Delta\alpha)} f(t,\alpha+\Delta\alpha)\,dt$$

$$\therefore \frac{\Delta\phi(\alpha)}{\Delta\alpha} = \int_{u(\alpha)}^{v(\alpha)} \boxed{\frac{f(t,\alpha+\Delta\alpha) - f(t,\alpha)}{\Delta\alpha}}\,dt - \frac{1}{\Delta\alpha}\boxed{\int_{u(\alpha)}^{u(\alpha+\Delta\alpha)} f(t,\alpha+\Delta\alpha)\,dt}$$

$$+ \boxed{\frac{1}{\Delta\alpha}\int_{v(\alpha)}^{v(\alpha+\Delta\alpha)} f(t,\alpha+\Delta\alpha)\,dt}$$

By MVT, $\exists \theta \in (\alpha, \alpha+\Delta\alpha)$ s.t.

(diff) $$\frac{f(t,\alpha+\Delta\alpha)-f(t,\alpha)}{\Delta\alpha} = \frac{\partial f(t,\theta)}{\partial\alpha}$$

By MVT (Thm 2.3), $\exists \xi_1$ s.t. $u(\alpha) < \xi_1 < u(\alpha+\Delta\alpha)$ s.t.

$$\int_{u(\alpha)}^{u(\alpha+\Delta\alpha)} f(t,\alpha+\Delta\alpha)\,dt = f(\xi_1,\alpha+\Delta\alpha)\left[u(\alpha+\Delta\alpha)-u(\alpha)\right]$$

$$\therefore \frac{1}{\Delta\alpha}\int_{u(\alpha)}^{u(\alpha+\Delta\alpha)} f(t,\alpha+\Delta\alpha)\,dt = f(\xi_1,\alpha+\Delta\alpha)\left[\frac{u(\alpha+\Delta\alpha)-u(\alpha)}{\Delta\alpha}\right]$$

Similary, MVT (2.3) $\exists \xi_2$ s.t. $v(\alpha) < \xi_2 < v(\alpha+\Delta\alpha)$ s.t.

$$\frac{1}{\Delta\alpha}\int_{v(\alpha)}^{v(\alpha+\Delta\alpha)} f(t,\alpha+\Delta\alpha)\,dt = f(\xi_2,\alpha+\Delta\alpha)\left[\frac{v(\alpha+\Delta\alpha)-v(\alpha)}{\Delta\alpha}\right]$$

Hence; $$\frac{\Delta\phi(\alpha)}{\Delta\alpha} = \int_{u(\alpha)}^{v(\alpha)} \frac{\partial f(t,\theta)}{\partial\alpha}\,dt - f(\xi_1,\alpha+\Delta\alpha)\left[\frac{u(\alpha+\Delta\alpha)-u(\alpha)}{\Delta\alpha}\right] + f(\xi_2,\alpha+\Delta\alpha)\left[\frac{v(\alpha+\Delta\alpha)-v(\alpha)}{\Delta\alpha}\right]$$

Take $\Delta\alpha \to 0$, we have $\theta \to \alpha$, $\xi_1 \to u(\alpha)$, $\xi_2 \to v(\alpha)$

$$\therefore \boxed{\frac{d}{d\alpha}\phi(\alpha) = \int_{u(\alpha)}^{v(\alpha)} \frac{\partial f(t,\alpha)}{\partial\alpha}\,dt - f(u,\alpha)\frac{du}{d\alpha} + f(v,\alpha)\frac{dv}{d\alpha}} \quad \#$$

In case that the lower limit and upper limit are constants;

i.e. $\phi(\alpha) = \int_a^b f(t,\alpha)\,dt$; a, b constants.

$$\therefore \frac{d}{d\alpha}\phi(\alpha) = \int_{u(\alpha)}^{v(\alpha)} \frac{\partial f(t,\alpha)}{\partial\alpha}\,dt + f(b,\alpha)\overset{0}{\frac{db}{d\alpha}} - f(a,\alpha)\overset{0}{\frac{da}{d\alpha}}$$

$$= \int_{u(\alpha)}^{v(\alpha)} \frac{\partial f(t,\alpha)}{\partial\alpha}\,dt$$

Define $F(x,y,z) = \int_y^z f(t,x)\,dt$, F_x, F_y, F_z exist.

$$\frac{\partial F}{\partial x} = \int_y^z \frac{\partial f(t,x)}{\partial x}\,dt + f(z,x)\frac{dz}{dx} - f(y,x)\frac{dy}{dx} = \int_y^z \frac{\partial f(t,x)}{\partial x}\,dt$$

$$\frac{\partial F}{\partial y} = \int_y^z \frac{\partial f(t,x)}{\partial y}\,dt + f(z,x)\frac{dz}{dy} - f(y,x)\frac{dy}{dy} = -f(y,x)$$

$$\frac{\partial F}{\partial z} = \int_y^z \frac{\partial f(t,x)}{\partial z}\,dt + f(z,x)\frac{dz}{dz} - f(y,x)\frac{dy}{dz} = f(z,x)$$

Ex If $\phi(\alpha) = \int_\alpha^{\alpha^2} \frac{\sin \alpha x}{x}\,dx$; $\alpha > 0$, Find $\phi'(\alpha)$

Sol^n let $f(x,\alpha) = \frac{\sin \alpha x}{x}$; $\alpha > 0$.

$$\therefore \phi'(\alpha) = \int_\alpha^{\alpha^2} \frac{\partial}{\partial \alpha} f(x,\alpha)\,dx + f(\alpha^2,\alpha)\frac{d\alpha^2}{d\alpha} - f(\alpha,\alpha)\frac{d\alpha}{d\alpha}$$

$$= \int_\alpha^{\alpha^2} \frac{(\cos \alpha x)\cdot \cancel{x}}{\cancel{x}}\,dx + \frac{\sin \alpha\cdot\alpha^2}{\alpha^{\cancel{2}}}(2\cancel{\alpha}) - \frac{\sin \alpha\cdot\alpha}{\alpha}$$

$$= \left.\frac{\sin \alpha x}{\alpha}\right|_{x=\alpha}^{\alpha^2} + \frac{2\sin \alpha^3}{\alpha} - \frac{\sin \alpha^2}{\alpha}$$

$$= \frac{\sin \alpha^3}{\alpha} - \frac{\sin \alpha^2}{\alpha} + \frac{2\sin \alpha^3}{\alpha} - \frac{\sin \alpha^2}{\alpha}$$

$$= \frac{3\sin \alpha^3}{\alpha} - \frac{2\sin \alpha^2}{\alpha} \qquad \#$$

Ex 2.15 If $\displaystyle\int_0^{\pi} \frac{dx}{\alpha - \cos x} = \frac{\pi}{\sqrt{\alpha^2-1}}$; $\alpha > 1$ Find $\displaystyle\int_0^{\pi} \frac{dx}{(2-\cos x)^2}$

Sol^{n} Let $\displaystyle \phi(\alpha) = \int_0^{\pi} \frac{dx}{\alpha - \cos x} = \frac{\pi}{\sqrt{\alpha^2-1}} = \pi(\alpha^2-1)^{-\frac{1}{2}}$

$$\therefore \phi'(\alpha) = \int_0^{\pi} \frac{\partial}{\partial\alpha}\left(\frac{1}{\alpha-\cos x}\right)dx = -\int_0^{\pi} \frac{1}{(\alpha-\cos x)^2}\,dx$$

However, $\displaystyle \phi'(\alpha) = -\frac{\pi}{2}(\alpha^2-1)^{-3/2}(2\alpha) = -\frac{\alpha\pi}{(\alpha^2-1)^{3/2}}$

$$\therefore \int_0^{\pi} \frac{1}{(\alpha-\cos x)^2}\,dx = \frac{\alpha\pi}{(\alpha^2-1)^{3/2}}$$

Set $\alpha = 2$

$$\therefore \int_0^{\pi} \frac{1}{(2-\cos x)^2}\,dx = \frac{2\pi}{3\sqrt{3}} \qquad \#$$

Thm 2.8 Let $\displaystyle \phi(\alpha) = \int_{u(\alpha)}^{v(\alpha)} f(t,\alpha)\,dt$, $f(t,\alpha)$ be cont in region s.t. $u \le t \le v$, $a \le \alpha \le b$. If u, v are constant, then

$$\int_a^b \phi(\alpha)\,d\alpha = \int_a^b \int_u^v f(t,\alpha)\,dt\,d\alpha = \int_u^v \int_a^b f(t,\alpha)\,d\alpha\,dt$$

Proof Assume that $\displaystyle F(\alpha) = \int_u^v \int_a^{\alpha} f(t,\alpha)\,d\alpha\,dt$

$$\therefore F'(\alpha) = \int_u^v \frac{\partial}{\partial\alpha}\int_a^{\alpha} f(t,\alpha)\,d\alpha\,dt = \int_u^v f(t,\alpha)\,dt = \phi(\alpha)$$

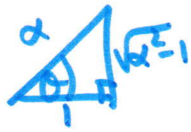
$$\therefore F(\alpha) = \int_a^{\alpha} \phi(\alpha)\,d\alpha + C \quad \text{s.t. } C \text{ is constant.}$$

$\because F(a) = 0$, Hence

$\displaystyle \int_u^v \int_a^{\alpha} f(t,\alpha)\,d\alpha\,dt = \int_a^{\alpha}\int_u^v F(t,\alpha)\,dt\,d\alpha$, let $\alpha = b$, we conclude the proof. #

Ex 2.16 Prove that for $a, b > 1$

$$\int_0^{\pi} \ln\left(\frac{b-\cos x}{a-\cos x}\right)dx = \pi \ln\left(\frac{b+\sqrt{b^2-1}}{a+\sqrt{a^2-1}}\right)$$



Proof From $\displaystyle\int_0^{\pi} \frac{dx}{\alpha-\cos x} = \frac{\pi}{\sqrt{\alpha^2-1}}$; $\alpha > 1$

ให้ $\alpha = \sec\theta$

$d\alpha = \sec\theta\tan\theta\, d\theta$

$$\therefore \int_a^b \int_0^{\pi} \frac{dx}{\alpha-\cos x}\, d\alpha = \int_a^b \frac{\pi}{\sqrt{\alpha^2-1}}\, d\alpha$$

$$\therefore \int \frac{\pi}{\sqrt{\alpha^2-1}}\, d\alpha = \pi\int \frac{\sec\theta \cancel{\tan\theta}\, d\theta}{\cancel{\sqrt{\sec^2\theta-1}}}$$

$$= \pi\int \sec\theta\, d\theta$$

$$= \pi\ln|\sec\theta+\tan\theta| + c$$

$$= \pi\ln|\alpha+\sqrt{\alpha^2-1}| + c$$

$$\int_0^{\pi}\int_a^b \frac{1}{\alpha-\cos x}\, d\alpha\, dx = \pi\left[\ln\left(\alpha+\sqrt{\alpha^2-1}\right)\right]_a^b$$

$$\int_0^{\pi} \ln(\alpha-\cos x)\Big|_a^b\, dx = \pi\ln\left(\frac{b+\sqrt{b^2-1}}{a+\sqrt{a^2-1}}\right)$$

$$\int_0^{\pi} \ln\left(\frac{b-\cos x}{a-\cos x}\right)dx = \pi\ln\left(\frac{b+\sqrt{b^2-1}}{a+\sqrt{a^2-1}}\right) \#.$$