Previous Tools

Theorem 3.1 ① Geometric or exponential integral.

$$\int_a^{+\infty} e^{-tx}\,dx \text{ s.t. } t \text{ is a constant, converges if } t>0, \text{ diverges if } t\le 0$$

② p-integral of the first kind

$$\int_a^{+\infty} \frac{1}{x^p}\,dx \text{ s.t. } p \text{ is a constant, converges if } p>1, \text{ diverges if } p\le 1$$

Theorem 3.2 ① $\int_a^b \frac{1}{(x-a)^p}\,dx$ converges if $p<1$ and diverges if $p\ge 1$.

② $\int_a^b \frac{1}{(b-x)^p}\,dx$ converges if $p<1$ and diverges if $p\ge 1$.

3.2.1 Tests for improper integral of the first kind.

Thm let $f(x)\ge 0$ for all $x\ge a$ and $\int_a^{+\infty} f(x)\,dx$ be the improper integral of the first kind.

Then $\int_a^{+\infty} f(x)\,dx$ converges $\iff \exists M>0$ s.t. $\int_a^t f(x)\,dx \le M \quad \forall t\ge a$

If $\int_a^{+\infty} f(x)\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx \le M$.

① Comparison Test. (for nonnegative integrand)

Thm 3.3 ① Convergence Let $g(x)\ge 0 \ \forall x\ge a$ and $\int_a^{+\infty} g(x)\,dx$ converges.

If $0\le f(x)\le g(x) \ \forall x\ge a$, then $\int_a^{+\infty} f(x)\,dx$ converges

② divergence let $g(x)\ge 0 \ \forall x\ge a$ and $\int_a^{+\infty} g(x)\,dx$ diverges.

If $g(x)\le f(x) \ \forall x\ge a$, then $\int_a^{+\infty} f(x)\,dx$ diverges.

② Quotient test

Thm 3.4 (1) If $f(x)\ge 0$ and $g(x)\ge 0$, and if $\lim_{x\to+\infty} \frac{f(x)}{g(x)} = A \ne 0$ or $+\infty$, then both $\int_a^{+\infty} f(x)\,dx$ and $\int_a^{+\infty} g(x)$ converges or diverges.

(2) If $A=0$ in (1), and $\int_a^{+\infty} g(x)\,dx$ converges, then $\int_a^{+\infty} f(x)\,dx$ converges

(3) If $A=+\infty$ in (1), and $\int_a^{+\infty} g(x)\,dx$ diverges, then $\int_a^{+\infty} f(x)\,dx$ diverges.

Thm 3.5 (Directed Results from Quotient test) Let $\lim_{x\to+\infty} x^p f(x) = A$. Then

(1) $\int_a^{+\infty} f(x)\,dx$ converges if $p>1$ and A is finite number.

(2) $\int_a^{+\infty} f(x)\,dx$ diverges if $p\le 1$ and $A\ne 0$ (A might be infinite number).

2 Absolute and Conditional Convergences

Definition: $\int_a^{+\infty} f(x)dx$ is s.t.b. absolutely convergent if $\int_a^{+\infty} |f(x)|dx$ converges.

If $\int_a^{+\infty} f(x)dx$ converges but $\int_a^{+\infty} |f(x)|dx$ diverges, $\int_a^{+\infty} f(x)dx$ is s.t.b. conditionally convergent.

Theorem 3.6 If $\int_a^{+\infty} |f(x)|dx$ converges, then $\int_a^{+\infty} f(x)dx$ converges.

3.2.2 Tests for improper integral of the second kind.

The following tests consider on the discontinuous point at $x=a$, when $a \le x \le b$. We can similarly perform the test in the case $x=b$, or $x=x_0$, where $a<x_0<b$.

① Comparison Test.

Theorem 3.7 ① Convergence: let $g(x)\geq 0$ for $a<x\le b$ and assume that $\int_a^b g(x)dx$ conv.

If $0\le f(x)\le g(x)$ for $a<x\le b$, then

$$\int_a^b f(x)dx \quad \text{converges.}$$

② Divergence: let $g(x)\geq 0$ for $a<x\le b$ and assume that $\int_a^b g(x)dx$ div.

If $f(x)\geq g(x)$ for $a<x\le b$, then

$$\int_a^b f(x)dx \quad \text{diverges.}$$

② Quotient test

Theorem 3.8 (1) If $f(x)\geq 0$, $g(x)\geq 0$ for $a<x\le b$, and $\lim_{x\to a} \frac{f(x)}{g(x)} = A \neq 0/\infty$,

Then both $\int_a^b f(x)dx$ and $\int_a^b g(x)dx$ converges or diverges.

(2) If $A=0$ in (1) and $\int_a^b g(x)dx$ converges, then $\int_a^b f(x)dx$ converges.

(3) If $A=+\infty$ in (1) and $\int_a^b g(x)dx$ diverges, then $\int_a^b f(x)dx$ diverges.

Theorem 3.9 (Directed Results from Quotient test) let $\lim_{x\to a^+} (x-a)^p f(x) = A$. Then

(1) $\int_a^b f(x)dx$ converges if $p<1$ and $A<\infty$.

(2) $\int_a^b f(x)dx$ diverges if $p\geq 1$ and $A\neq 0$.

Theorem 3.10 (Directed Results from Quotient test) let $\lim_{x\to b^-} (x-a)^p f(x) = B$ Then

(1) $\int_a^b f(x)dx$ converges if $p<1$ and $B<\infty$.

(2) $\int_a^b f(x)dx$ diverges if $p\geq 1$ and $B\neq 0$.

3.2 Some theorems and Test for convergence.

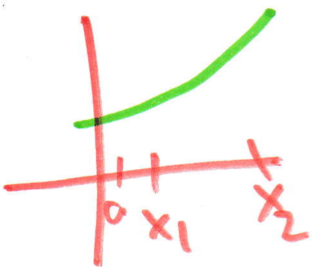
3.2.1 Tests for improper integral of the first kind

Nonnegative function case

Let f be an nonnegative $f^{\underline{n}}$ on $[a,\infty)$ which is integrable on $[a,t]$ $\forall t \geq a$.

Define $F(t) = \int_a^t f(x)\,dx$ For $t \in [a,\infty)$

F is nondecreasing function on $[a,\infty)$, $F(a)=0$



$a \leq x_1 < x_2 \Rightarrow F(x_1) \leq F(x_2)$

$$\cancel{F(x_2) - F(x_1)} = \int_a^{x_2} f(x)\,dx - \int_a^{x_1} f(x)\,dx$$

$$= \int_{x_1}^{x_2} f(x)\,dx \geq 0$$

$\therefore F(x_2) \geq F(x_1)$

Hence, $\lim\limits_{t\to+\infty} F(t)$ is positive number / infinity.

"Monotonic Sequence Theorem"

If f has upper bound, $F(t)$ converges when $t\to\infty$

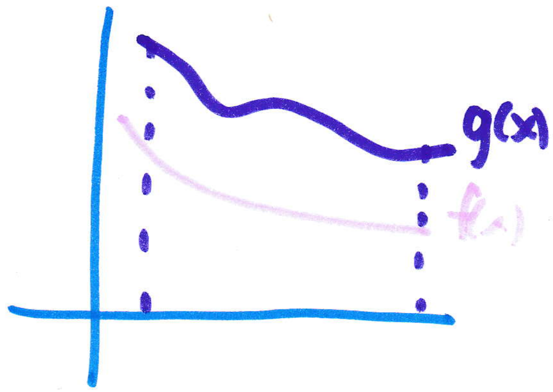
If f does not have upper bd, $F(t) \to \infty$ when $t\to\infty$

$\therefore$ for $f:[a,\infty) \to [0,\infty)$ which is integrable on $[a,t]$ $\forall t \geq a$;

$$\int_a^{\infty} f(x)\,dx \text{ converges} \Leftrightarrow \lim_{x\to\infty} F(x) \text{ exist.}$$

① Comparison Test

$$\boxed{\int_a^{+\infty} f(x)\,dx} \xrightarrow{921} \int_a^{+\infty} g(x)\,dx$$



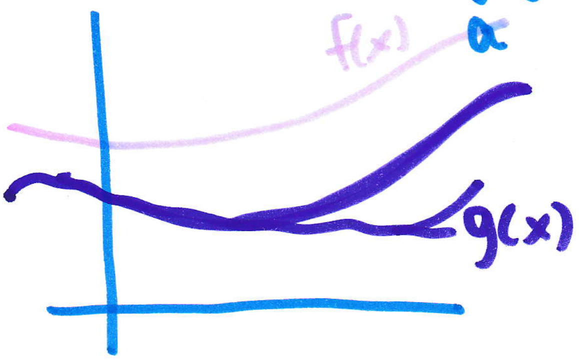
$g(x) \geq 0$

$\boxed{1}$ $0 \leq f(x) \leq g(x)$

If $\int_a^{+\infty} g(x)\,dx$ converges, $\int_a^{+\infty} f(x)\,dx$ converges

$\boxed{2}$ ~~If~~ $g(x) \leq f(x)$

If $\int_a^{+\infty} g(x)\,dx$ diverges, $\int_a^{+\infty} f(x)\,dx$ diverges.



Proof ① Assume that $0 \leq f(x) \leq g(x)$ and $\int_a^{+\infty} g(x)$ converges

$$\therefore \int_a^{+\infty} f(x)\,dx = \lim_{b\to+\infty} \int_a^b f(x)\,dx \leq \lim_{b\to+\infty} \int_a^b g(x)\,dx$$

$$= \int_a^{+\infty} g(x)\,dx < M$$

$$\therefore \int_a^{+\infty} f(x)\,dx \text{ converges.}$$

Ex $\displaystyle\int_1^{+\infty} \frac{dx}{\sqrt{x^2+1}}$ From $4x^2 \geq x^2+1 \quad \forall x \geq 1$

$$2x \geq \sqrt{x^2+1}$$

$$\frac{1}{\sqrt{x^2+1}} \geq \frac{1}{2x}$$

Consider $\displaystyle\int_1^{+\infty} \frac{1}{2x}\,dx$ diverges (Thm 3.1 2)

[$\displaystyle\int_1^{+\infty} \frac{1}{2x}\,dx$ is p-integral s.t. $p = 1$
By Thm 3.1 2, $\displaystyle\int_1^{+\infty} \frac{1}{2x}\,dx$ diverges]

Hence, $\displaystyle\int_1^{+\infty} \frac{dx}{\sqrt{x^2+1}}$ diverges by Comparison Test.

$$\Downarrow$$

$$\int_0^{+\infty} \frac{dx}{\sqrt{x^2+1}}$$

diverges.

Ex $\displaystyle\int_a^{+\infty} \frac{1}{e^x+1}\,dx$ $\qquad e^x+1 \geq e^x \Rightarrow \frac{1}{e^x} \geq \frac{1}{e^x+1}$

From $\displaystyle\int_a^{+\infty} \frac{1}{e^x}\,dx$ converses (Thm 3.1 1),

$\displaystyle\int_a^{+\infty} \frac{1}{e^x+1}\,dx$ converse (by Comparison Test)

Ex 3.7 $\displaystyle\int_2^{+\infty} \frac{1}{\ln x}\,dx$ diverges $\nearrow$ $x \geq \ln x$, $\quad \dfrac{1}{\ln x} \geq \dfrac{1}{x}$ $\quad \displaystyle\int_2^{+\infty} \frac{1}{x}\,dx$ div.

② Quotient Test

Thm 3.4 **Proof** Suppose that $\boxed{\lim_{x\to+\infty}\frac{f(x)}{g(x)}=A} > 0$

$\therefore\ \forall\varepsilon>0\ \exists N$ s.t. for all $x \geq N$

$$\left|\frac{f(x)}{g(x)}-A\right|<\varepsilon.$$

Hence, $$-\varepsilon+A<\frac{f(x)}{g(x)}<\varepsilon+A$$

$$\Longleftrightarrow (A-\varepsilon)g(x) < f(x) < (A+\varepsilon)g(x)$$

$$\Longleftrightarrow (A-\varepsilon)\int_N^b g(x)dx < \int_N^b f(x)dx < (A+\varepsilon)\int_N^b g(x)dx. \quad (*)$$

Choose $\varepsilon < A \Rightarrow A-\varepsilon>0$

Take ~~lim~~ $b\to+\infty$

$(*)$; $$(A-\varepsilon)\int_N^{+\infty} g(x)dx < \int_N^{+\infty} f(x)dx < (A+\varepsilon)\int_N^{+\infty} g(x)dx$$

Remark that $$\int_a^{+\infty} g(x)dx = \int_a^N g(x)dx + \int_N^{+\infty} g(x)dx$$

$$\int_a^{+\infty} f(x)dx = \int_a^N f(x)dx + \int_N^{+\infty} f(x)dx$$

$$(A-\varepsilon)\int_a^{+\infty} g(x)dx < \int_a^{+\infty} f(x)dx < (A+\varepsilon)\int_a^{+\infty} g(x)dx$$

1. $\int_a^{+\infty} g(x)dx$ converges $\Rightarrow \int_a^{+\infty} f(x)dx$ conv.

2. $\int_a^{+\infty} g(x)dx$ diverges $\Rightarrow \int_a^{+\infty} f(x)dx$ div.

#

Ex $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt{x^2+1}}\,dx$ when $x\to\infty$, $\dfrac{1}{\sqrt{x^2+1}} \approx \dfrac{1}{\sqrt{x^2}} = \dfrac{1}{x}$

$f(x) = \dfrac{1}{\sqrt{x^2+1}}$, $g(x) = \dfrac{1}{x}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{1}{\sqrt{x^2+1}}\right) \Big/ \left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}} = 1$$

Since $\displaystyle\int_0^{+\infty} \frac{1}{x}\,dx$ diverges

$\therefore \displaystyle\int_0^{+\infty} \frac{1}{\sqrt{x^2+1}}\,dx$ diverges. #

Ex $\displaystyle\int_1^{+\infty} \frac{x-1}{x^2+x-1}\,dx$ $f(x) = \dfrac{x-1}{x^2+x-1}$ cont on $[1,\infty)$

$= \left[x^2 + 2(x)\left(\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right)^2\right] - \tfrac{1}{4} - 1$

$= \left(x + \tfrac{1}{2}\right)^2 - \tfrac{5}{4}$

$\cancel{g(x)} = \dfrac{1-\frac{1}{x}}{x+1-\frac{1}{x}} = \dfrac{1}{x}$ when $x\to\infty$

$g(x) = \dfrac{1}{x}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \left(\frac{x-1}{x^2+x-1}\right) \Big/ \left(\frac{1}{x}\right)$$

$$= \lim_{x\to\infty} \frac{x^2-x}{x^2+x-1} = 1$$

Since $\displaystyle\int_1^{+\infty} \frac{1}{x}\,dx$ diverges, $\displaystyle\int_1^{+\infty} \frac{x-1}{x^2+x-1}\,dx$ diverges.

$$\int_0^{+\infty} \frac{x^2}{5x^4+x+1}\,dx \qquad g(x)=\frac{1}{x^2}$$

$$\lim_{x\to+\infty} \left(\frac{x^2}{5x^4+x+1}\right) \Big/ \left(\frac{1}{x^2}\right) = \lim_{x\to+\infty} \frac{x^4}{5x^4+x+1} = \frac{1}{5}$$

Consider $\int_0^{+\infty} \frac{1}{x^2}\,dx$ converges.

Hence, $\int_0^{+\infty} \frac{x^2}{5x^4+x+1}\,dx$ converges #

Ex $\int_1^{+\infty} \frac{x^2}{x^4+x^2-2}\,dx$ เลือก $g(x)=\frac{1}{x^2}$

$$\lim_{x\to+\infty} \left(\frac{x^2}{x^4+x^2-2}\right) \Big/ \frac{1}{x^2} = 1 \qquad \int_1^{+\infty} \frac{1}{x^2}\,dx \text{ conv.}$$

$\therefore \int_1^{+\infty} \frac{x^2}{x^4+x^2-2}\,dx$ converges.

Absolute and Conditional Convergences

Ex $\int_0^{+\infty} \frac{\cos x}{x^2+1}\,dx$

Consider $\int_0^{+\infty} \left|\frac{\cos x}{x^2+1}\right| dx \le \int_0^{+\infty} \frac{1}{x^2+1}\,dx$

$$\lim_{x\to+\infty} \frac{f(x)}{g(x)} = \lim_{x\to+\infty} \left(\frac{1}{x^2+1}\right) \Big/ \underbrace{\frac{1}{x^2}}_{g(x)} = 1.$$

$\int_0^{+\infty} \frac{1}{x^2}\,dx$ conv. $\Rightarrow \int_0^{+\infty} \frac{1}{x^2+1}\,dx$ conv $\Rightarrow \int_0^{+\infty} \left|\frac{\cos x}{x^2+1}\right| dx$ conv.

By Thm 3.6., $\int_0^{+\infty} \frac{\cos x}{x^2+1}\,dx$ conv. #

<u>Remark</u> Thm 3.6., converse is not always true.

<u>Ex</u> $\int_1^{+\infty} f(x)\,dx$ where $f(x) = \begin{cases} \frac{1}{x} & ; x \in \mathbb{Q} \\ -\frac{1}{x} & ; x \notin \mathbb{Q} \end{cases}$

<u>Ex</u> $\int_1^{+\infty} \sin\left(\frac{3}{x}\right) dx$; $\sin\left(\frac{3}{x}\right) > 0 \quad \forall x \geq 1$

(ติดอ)

$$\lim_{x \to +\infty} \frac{\sin\left(\frac{3}{x}\right)}{\frac{1}{x}} = 3$$

$$\therefore \int_1^{+\infty} \frac{1}{x}\,dx \text{ div} \Rightarrow \int_1^{+\infty} \sin\left(\frac{3}{x}\right) dx \text{ div.}$$