⊛ Improper integral of the second kind (continue).

Absolute and Conditional Convergence.

Def^{n} : $\int_a^b f(x)\,dx$ is s.t.b. absolutely convergent if $\int_a^b |f(x)|\,dx$ converges.

If $\int_a^b f(x)\,dx$ converges but $\int_a^b |f(x)|\,dx$ diverges, $\int_a^b f(x)\,dx$ is conditionally convergent.

Thm 3.11 If $\int_a^b |f(x)|\,dx$ converges, then $\int_a^b f(x)\,dx$ converges.

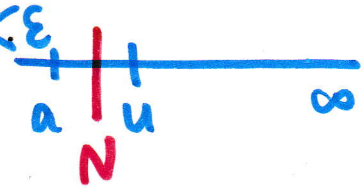
3.2.3 Improper integral with parameters and uniformly convergence.

[related to infinity : improper int, Type 1)

let $\phi(\alpha) = \int_a^{+\infty} f(x,\alpha)\,dx$ such that $\alpha_1 \le \alpha \le \alpha_2$. We define the concept of uniformly convergence by the following definition.

Def^{n} 3.4 $\phi(\alpha) = \int_a^{+\infty} f(x,\alpha)\,dx$ is said to be uniformly convergent for all $\alpha_1 \le \alpha \le \alpha_2$ if and only if $\forall \varepsilon > 0\ \exists N_\varepsilon$ s.t. for every $u > N$ and $\alpha \in [\alpha_1, \alpha_2]$,

$$\left| \phi(\alpha) - \int_a^u f(x,\alpha)\,dx \right| = \left| \int_u^{+\infty} f(x,\alpha)\,dx \right| < \varepsilon$$



[Meaning]

Tests for uniform convergence.

① Weierstrass M-Test

Theorem 3.12 If we can find a function $M(x) \geq 0$ s.t.

(1) $|f(x,\alpha)| \le M(x)$; for all $\alpha_1 \le \alpha \le \alpha_2$ and $x > a$, and

(2) $\int_a^{+\infty} M(x)\,dx$ converges,

Then $\int_a^{+\infty} f(x,\alpha)\,dx$ converges uniformly and absolutely in $[\alpha_1, \alpha_2]$

② Dirichlet's Test

Theorem 3.13 Let (1) $\psi(x)$ is monotone decreasing s.t. $\lim_{x \to +\infty} \psi(x) = 0$, and

(2) $\left| \int_a^u f(x,\alpha)\,dx \right| < P$ for all $u > a$ and $\alpha_1 \le \alpha \le \alpha_2$,

Then $\int_a^{+\infty} f(x,\alpha)\psi(x)\,dx$ converges uniformly for all $\alpha_1 \le \alpha \le \alpha_2$.

2

3.2.4 Theorems of uniform convergence.

Theorem 3.14 If $f(x,\alpha)$ is continuous for $x \geq a$ and $\alpha_1 \leq \alpha \leq \alpha_2$, and if $\int_a^{+\infty} f(x,\alpha)dx$ is $\boxed{\text{uniformly convergent}}$ for $\alpha_1 \leq \alpha \leq \alpha_2$,

Then $\phi(\alpha) = \int_a^{+\infty} f(x,\alpha)\,dx$ is continuous in $\alpha_1 \leq \alpha \leq \alpha_2$.

Especially, if $\alpha_0 \in [\alpha_1, \alpha_2]$, then

$$\lim_{\alpha\to\alpha_0} \phi(\alpha) = \lim_{\alpha\to\alpha_0} \int_0^{+\infty} f(x,\alpha)dx = \int_a^{+\infty} \lim_{\alpha\to\alpha_0} f(x,\alpha)\,d\alpha. \qquad (*)$$

Theorem 3.15 Under the conditions of Thm 3.14, then

$$\int_{\alpha_1}^{\alpha_2} \phi(\alpha)d\alpha = \int_{\alpha_1}^{\alpha_2}\left(\int_a^{+\infty} f(x,\alpha)dx\right)d\alpha = \int_a^{+\infty}\left(\int_{\alpha_1}^{\alpha_2} f(x,\alpha)d\alpha\right)dx$$

which corresponds to a change of order of integration.

Theorem 3.16 Let $f(x,\alpha)$ be continuous and has a continuous partial derivative w.r.t. α for $x \geq a$ and $\alpha_1 \leq \alpha \leq \alpha_2$, and if $\int_a^{+\infty} \frac{\partial f}{\partial \alpha}dx$ converges uniformly in $\alpha_1 \leq \alpha \leq \alpha_2$, then if a does not depend on α,

$$\frac{d\phi}{d\alpha} = \int_a^{+\infty} \frac{\partial f}{\partial \alpha}\,dx.$$

(~ Leibnitz Thm)

(If a depends on α, the result is easily modified!)

PS By Calculus 206111, we know the important result:

> **1.5.5 THEOREM** *If* $\lim_{x\to c} g(x) = L$ *and if the function* f *is continuous at* L, *then* $\lim_{x\to c} f(g(x)) = f(L)$. *That is,*
>
> $$\lim_{x\to c} f(g(x)) = f\left(\lim_{x\to c} g(x)\right)$$
>
> *This equality remains valid if* $\lim_{x\to c}$ *is replaced everywhere by one of* $\lim_{x\to c^+}$, $\lim_{x\to c^-}$, $\lim_{x\to+\infty}$, *or* $\lim_{x\to-\infty}$.

Tests for improper integral of the second kind.

Ex ① $\displaystyle\int_1^5 \frac{1}{\sqrt{x^4-1}}\,dx$

Since $x > 1$; $x^4 - 1 > x - 1 \Rightarrow \dfrac{1}{x-1} > \dfrac{1}{x^4-1}$

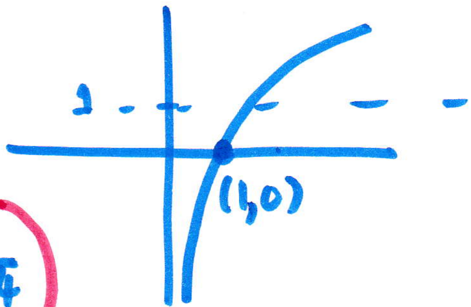
$$\frac{1}{\sqrt{x^4-1}} < \frac{1}{\sqrt{x-1}}$$

$\displaystyle\int_1^5 \frac{1}{\sqrt{x-1}}\,dx$ conv/div Thm 3.2 ① $a=1$ $p=\frac{1}{2}$

$\therefore \displaystyle\int_1^5 \frac{1}{\sqrt{x-1}}\,dx$ converges

$\therefore \displaystyle\int_1^5 \frac{1}{\sqrt{x^4-1}}\,dx$ converges. by Comparison Test #

Ex ② $\displaystyle\int_3^6 \frac{\ln x}{(x-3)^4}\,dx$



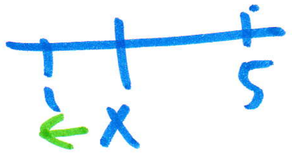
Since $x > 3$, $\dfrac{\ln x}{(x-3)^4} > \dfrac{1}{(x-3)^4}$

$\displaystyle\int_3^6 \frac{1}{(x-3)^4}\,dx$ $p=4$; $a=3$ Thm 3.2 ①

$\therefore \displaystyle\int_3^6 \frac{1}{(x-3)^4}\,dx$ diverges.

$\therefore \displaystyle\int_3^6 \frac{\ln x}{(x-3)^4}\,dx$ diverges. by Comparison Test #

Ex $\displaystyle\int_1^5 \underbrace{\frac{1}{\sqrt{x^4-1}}}_{f(x)}\,dx$ $\qquad \left[\lim_{x\to 1^+} \frac{f(x)}{g(x)} = A\right]$



Choose $g(x) = \dfrac{1}{\sqrt{x-1}}$

$$\lim_{x\to 1^+} \left(\frac{1}{\sqrt{x^4-1}}\right) \Big/ \frac{1}{\sqrt{x-1}} = \lim_{x\to 1^+} \sqrt{\frac{x-1}{x^4-1}}$$

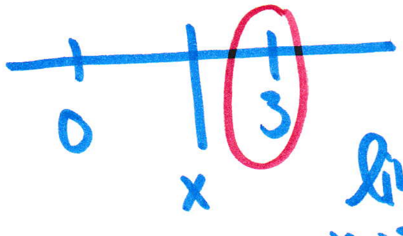
$$= \lim_{x\to 1^+} \sqrt{\frac{\cancel{x-1}}{(x^2+1)\cancel{(x-1)}(x+1)}}$$

$$= \frac{1}{2} \qquad \#$$

Since $\displaystyle\int_1^5 \frac{1}{\sqrt{x-1}}\,dx$ conv, (Thm 3.2 $a=1, p=\frac{1}{2}$)

~~then~~ $\displaystyle\int_1^5 \frac{1}{\sqrt{x^4-1}}\,dx$ conv. by Quotient test.

Ex $\displaystyle\int_0^3 \underbrace{\frac{1}{(3-x)\sqrt{x^2+1}}}_{f(x)}\,dx$ $\qquad \left[\lim_{x\to 3^-} \frac{f(x)}{g(x)} = A\right]$



choose $g(x) = \dfrac{1}{3-x}$

$$\lim_{x\to 3^-} \left(\frac{1}{(3-x)\sqrt{x^2+1}}\right) \Big/ \frac{1}{3-x} = \lim_{x\to 3^-} \frac{\cancel{3-x}}{\cancel{(3-x)}\sqrt{x^2+1}}$$

$$= \frac{1}{\sqrt{10}}$$

Consider $\displaystyle\int_0^3 \frac{1}{3-x}\,dx$, by Thm 3.2 ② ($a=3, p=1$),

$\displaystyle\int_0^3 \frac{1}{3-x}\,dx$ div.

$\therefore \displaystyle\int_0^3 \frac{1}{(3-x)\sqrt{x^2+1}}\,dx$ diverges. by Quotient test. #

Ex Verifying whether $\int_{\pi}^{4\pi} \frac{\sin x}{\sqrt[3]{x-\pi}}\,dx$ converges or not.

Sol^{n} Remark that $x-\pi \geq 0$ for $x \in [\pi, 4\pi]$

$(x-\pi > 0$ for $x \in (\pi, 4\pi])$

(Show that $\int_{\pi}^{4\pi} \left|\frac{\sin x}{\sqrt[3]{x-\pi}}\right| dx$ converges or not)

Consider $\left|\frac{\sin x}{\sqrt[3]{x-\pi}}\right| \leq \frac{1}{\sqrt[3]{x-\pi}}$

Then $\int_{\pi}^{4\pi} \frac{1}{\sqrt[3]{x-\pi}}\,dx$ converges by Thm 3.2 $(a=\pi, p=\frac{1}{3})$.

Hence, $\int_{\pi}^{4\pi} \left|\frac{\sin x}{\sqrt[3]{x-\pi}}\right| dx$ converges.

By Thm 3.11, $\int_{\pi}^{4\pi} \frac{\sin x}{\sqrt[3]{x-\pi}}\,dx$ converges. #

3.2.3 Improper integral with parameters and uniformly convergence.

Ex $F(x)$: $f(s) = \mathcal{L}\{F(x)\} = \int_{0}^{+\infty} e^{-sx} F(x)\,dx$

Ex $\phi(\alpha) = \int_{0}^{+\infty} f(x,\alpha)\,dx \longrightarrow 1^{st}$ improper integral

s.t. $f(x,\alpha)$ is function of two variables s.t.

$\alpha \in [\alpha_1, \alpha_2]$, $x \in [a, \infty)$

① $\phi(\alpha)$ cont? ② $\phi(\alpha)$ diff? ③ $\phi(\alpha)$ integrable?

Proof Thm 3.12 (Weierstrass M-Test)

Let $\varepsilon>0$. Since $\int_{0}^{+\infty} M(x)dx$ conv.

$\exists N_\varepsilon$ s.t. for every $u>N_\varepsilon$, $\alpha\in[\alpha_1,\alpha_2]$

$$\left|\int_a^{+\infty} M(x)dx - \int_a^u M(x)dx\right| = \left|\int_u^{+\infty} M(x)dx\right| < \varepsilon$$

$$\therefore \left|\int_a^{+\infty} f(x,\alpha)dx - \int_a^u f(x,\alpha)dx\right| = \left|\int_u^{+\infty} f(x,\alpha)dx\right|$$

$$\leq \int_u^{+\infty} |f(x,\alpha)|dx$$

$$\leq \int_u^{+\infty} M(x)dx < \varepsilon. \ \#$$

Ex Verifying whether $\int_0^{+\infty} \frac{\cos \alpha x}{x^2+1}dx$ is uniformly conv.? absolutely conv.?

(Weierstrass M-Test : นา $M(x)$ s.t.
$\left|\frac{\cos\alpha x}{x^2+1}\right| \leq M(x)$, $\int_0^{+\infty} M(x)dx$ conv.)

Consider $\left|\frac{\cos \alpha x}{x^2+1}\right| \leq \overbrace{\frac{1}{x^2+1}}^{M(x)}$ for all $\alpha\in\mathbb{R}$, $x>0$

Also, $\int_0^{+\infty} M(x)dx = \int_0^{+\infty} \frac{1}{x^2+1}dx$ conv. (Quotient test
$g(x)=\frac{1}{x^2}$; $\lim_{x\to\infty}\frac{f(x)}{g(x)} = 1$
and $\int_0^{+\infty}\frac{1}{x^2}dx$ conv.)

Hence, $\int_0^{+\infty}\frac{\cos\alpha x}{x^2+1}dx$ conv. uniformly + absolutely.
by Weierstrass M-Test #

Weierstraß

Exercise 3.2(7)

$$\lim_{\alpha\to0^+}\int_0^{+\infty}\alpha e^{-\alpha x}\,dx \overset{?}{=} \int_0^{+\infty}\left(\lim_{\alpha\to0^+}\alpha e^{-\alpha x}\right)dx \quad (\alpha>0)\ \boxed{5}$$

① $$\underbrace{\lim_{\alpha\to0^+}\int_0^{+\infty}\alpha e^{-\alpha x}\,dx}_{?!?} \quad \text{VS} \quad \underbrace{\int_0^{+\infty}\underbrace{\left(\lim_{\alpha\to0^+}\alpha e^{-\alpha x}\right)}_{0}dx}_{=0}$$

② Compute $\phi(\alpha)=\int_0^{+\infty}\alpha e^{-\alpha x}\,dx$ for $\alpha>0$

③ $\boxed{1}$ is uniformly convergent to 1 for $\alpha\geqslant\alpha_1>0$

④ $\boxed{1}$ is not uniformly convergent to 1 for $\alpha>1$