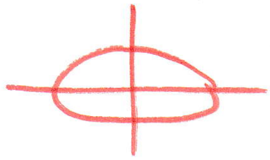
206331 : Advanced Calculus

3.4 Elliptic Integral : Lergendre's form & Jacobi's form.



3.4.1 Lergendre's elliptic Integral.

① Lergendre's elliptic integral of the first kind.

Define $$u = F(k,\phi) = \int_0^{\phi} \frac{1}{\sqrt{1-k^2\sin^2\theta}}\, d\theta \; ; \quad 0<|k|<1 \quad \text{—①}$$

s.t. ϕ is s.t.b. amplitude of u. i.e. $\phi = \text{amp } u$.

k is s.t.b. modulus of u; i.e. $k = \text{mod } u$.

If k is set to be arbitrary, ① is s.t.b. incomplete elliptic integral of the first kind.

If $\phi = \frac{\pi}{2}$, u is written as

$$K = K(k) = \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\theta}}\, d\theta \; ; \quad 0<|k|<1. \quad \text{—}\boxed{1.1}$$

and s.t.b. complete elliptic integral of the first kind.

② Lergendre's elliptic integral of the second kind.

Define $$E(k,\phi) = \int_0^{\phi} \sqrt{1-k^2\sin^2\theta}\, d\theta \quad ; \quad 0<|k|<1 \quad \text{—②}$$

If $\phi = \frac{\pi}{2}$, $E(k,\frac{\pi}{2})$ is

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\, d\theta \quad ; \quad 0<|k|<1 \quad \text{—}\boxed{2.1}$$

② is s.t.b. incomplete elliptic integral of the second kind.

$\boxed{2.1}$ is s.t.b. complete elliptic integral of the second kind.

③ Lergendre's elliptic integral of the third kind.

Define $$\pi(k,n,\phi) = \int_0^{\phi} \frac{1}{(1+n\sin^2\theta)\sqrt{1-k^2\sin^2\phi}}\, d\theta \quad ; \quad 0<|k|<1. \quad \text{—③}$$

s.t. n is a constant s.t. $n \neq 0$ [if $n=0$, it implies to be the first kind]

③ is s.t.b. incomplete elliptic integral of the third kind.

If $\phi = \frac{\pi}{2}$, ③ is s.t.b. complete elliptic integral of the third kind.

3.4.2 Jacobi's elliptic integral transform $v = \sin\theta$, let $x = \sin\phi$

① Jacobi's elliptic integral of the first kind.

Let $v = \sin\theta$, $x = \sin\phi$ $\Rightarrow \theta = \arcsin v$ $\Rightarrow d\theta = \frac{1}{\sqrt{1-v^2}} dv$

if $\theta = 0$, $v = \sin 0 = 0$, if $\theta = \phi$, $v = \sin\phi = x$

$$\therefore \; F_1(k,x) = \int_0^{x} \frac{1}{\sqrt{1-v^2}(1-k^2v^2)}\, d\theta \quad ; \quad 0<|k|<1$$

If $\phi = \frac{\pi}{2}$, then $x = \sin\frac{\pi}{2} = 1$. i.e.

$$F_1(k,1) = \int_0^{1} \frac{1}{\sqrt{1-v^2}\,(1-k^2v^2)}\, d\theta \quad ; \quad 0<|k|<1.$$

which is s.t.b. Jacobi complete elliptic integral of the first kind.

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② Jacobi elliptic integral of the second kind

By transforming $v = \sin\theta$, $x = \sin\phi$, we have

$$E_1(k,x) = \int_0^x \sqrt{\frac{1-k^2v^2}{1-v^2}}\,dv \quad ; \quad 0<|k|<1$$

If $\phi = \frac{\pi}{2}$, $x = \sin\frac{\pi}{2} = 1$, then

$$E_1(k,1) = \int_0^1 \sqrt{\frac{1-k^2v^2}{1-v^2}}\,dv \quad ; \quad 0<|k|<1$$

Which is s.t.b. Jacobi's complete elliptic integral of the second kind.

③ Jacobi's elliptic integral of the third kind.

By transforming $v = \sin\theta$, $x = \sin\phi$, then

$$\Pi_1(k,n,x) = \int_0^x \frac{1}{(1+nv^2)\sqrt{(1-v^2)(1-k^2v^2)}}\,dv \; ; \; 0<|k|<1$$

If $\phi = \frac{\pi}{2}$, $x = \sin\frac{\pi}{2} = 1$, then

$$\Pi_1(k,n,1) = \int_0^1 \frac{1}{(1+nv^2)\sqrt{(1-v^2)(1-k^2v^2)}}\,dv \; ; \; 0<|k|<1$$

Which is s.t.b. Jacobi's complete elliptic integral of the third kind.

Consider $\int R(x,y)\,dx$. We can transform $R(x,y)$ to the elementary function.

Ex

$$\left.\begin{aligned} y &= \sqrt{ax+b} \\ y &= \sqrt{ax^2+bx+c} \\ y &= \sqrt{ax^3+bx^2+cx+d} \\ y &= \sqrt{ax^4+bx^3+cx^2+dx+e} \end{aligned}\right\} \int R(x,y)\,dx \text{ is s.t.b. integral reducible to elliptic type.}$$

We define elliptic function by $[x = \sin\phi]$

$$\left.\begin{aligned} x &= \sin(\operatorname{amp} u) \equiv \operatorname{sn} u \\ \sqrt{1-x^2} &= \cos(\operatorname{amp} u) \equiv \operatorname{cn} u \\ \sqrt{1-k^2x^2} &= \sqrt{1-k^2\sin^2 u} \equiv \operatorname{dn} u \\ \frac{x}{\sqrt{1-x^2}} &= \frac{\operatorname{sn} u}{\operatorname{cn} u} \equiv \operatorname{tn} u \end{aligned}\right\} \quad \text{s.t.} \quad \begin{aligned} &\operatorname{sn}^2 u + \operatorname{cn}^2 u = 1 \\ &\operatorname{tn} u = \frac{\operatorname{sn} u}{\operatorname{cn} u}. \end{aligned}$$

Also

$$\frac{d}{du}(\operatorname{sn} u) = (\operatorname{cn} u)(\operatorname{dn} u) \qquad \operatorname{sn}(-u) = -\operatorname{sn} u$$

$$\frac{d}{du}(\operatorname{cn} u) = -(\operatorname{sn} u)(\operatorname{dn} u) \qquad \operatorname{cn}(-u) = \operatorname{cn} u$$

$$\frac{d}{du}(\operatorname{dn} u) = -k^2(\operatorname{sn} u)(\operatorname{dn} u) \qquad \operatorname{dn}(-u) = \operatorname{dn} u$$

$$\frac{d}{du}(\operatorname{tn} u) = \frac{\operatorname{dn} u}{\operatorname{cn}^2 u} \qquad \operatorname{tn}(-u) = -\operatorname{tn} u.$$

Trigonometric substitution

① $\sqrt{a^2-u^2}$, a^2-u^2 $\Rightarrow$ $u = a\sin\theta$; $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

② $\sqrt{a^2+u^2}$, a^2+u^2 $\Rightarrow$ $u = a\tan\theta$; $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$

③ $\sqrt{u^2-a^2}$, u^2-a^2 $\Rightarrow$ $u = a\sec\theta$; $0<\theta<\frac{\pi}{2}$ or $\pi<\theta<\frac{3\pi}{2}$.

Ex 3.20 ① Show that $\phi(\alpha)=\int_0^{+\infty}\frac{1}{x^2+\alpha}dx$ is uniformly conv. (for $\alpha \geqslant 1$)

② Show that $\phi(\alpha)=\frac{\pi}{2\sqrt{\alpha}}$.

③ Evalute $\int_0^{+\infty}\frac{1}{(x^2+1)^2}dx$

Sol$^{\underline{n}}$ ① choose $f(x,\alpha)=\frac{1}{x^2+\alpha}$ for $\alpha \geqslant 1$

By Weierstrass M-test,

Remark that $x^2+1 \leq x^2+\alpha \Rightarrow \frac{1}{x^2+\alpha} \leq \frac{1}{x^2+1}$

$$\left[\,|f(x,\alpha)| \leq \heartsuit \;,\; \int_0^{+\infty} \heartsuit \, d\sim \text{ conv.}\right]$$

$$\therefore \left|\frac{1}{x^2+\alpha}\right| \leq \frac{1}{x^2+1}$$

Consider $\int_0^{+\infty}\frac{1}{x^2+1}dx$ converges.

By Weierstrass M-test, $\int_0^{+\infty}\frac{1}{x^2+\alpha}dx$ converges uniformly.

② $\phi(\alpha)=\frac{\pi}{2\sqrt{\alpha}}$

$$\phi(\alpha)=\int_0^{+\infty}\frac{1}{x^2+\alpha}dx = \lim_{b\to+\infty}\int_0^{b}\frac{1}{x^2+\alpha}dx$$

$$=\lim_{b\to+\infty}\left[\frac{1}{\sqrt{\alpha}}\arctan\left(\frac{x}{\sqrt{\alpha}}\right)\right]_0^b$$

$$=\lim_{b\to+\infty}\left[\frac{1}{\sqrt{\alpha}}\arctan\left(\frac{b}{\sqrt{\alpha}}\right)-\frac{1}{\sqrt{\alpha}}\underbrace{\arctan 0}_{0}\right]$$

$$=\left(\frac{1}{\sqrt{\alpha}}\right)\left(\frac{\pi}{2}\right)=\frac{\pi}{2\sqrt{\alpha}}$$

③ Evaluate $\displaystyle\int_0^{+\infty} \frac{1}{(x^2+1)^2}\,dx$

<u>In general</u>, consider $\displaystyle\int_0^{+\infty} \frac{1}{(x^2+\alpha)^2}\,dx$

$$\left(\int_0^{+\infty} \frac{1}{(x^2+\alpha)}\,dx = \frac{\pi}{2\sqrt{\alpha}} \right)$$

Firstly, we show that $\displaystyle\int_0^{+\infty} \frac{1}{(x^2+\alpha)^2}\,dx$ unif. conv. for $\alpha \geq 1$.

Since $\displaystyle\left|\frac{1}{(x^2+\alpha)^2}\right| \leq \frac{1}{x^2+\alpha} \leq \frac{1}{x^2+1}$ s.t.

$\displaystyle\int_0^{+\infty} \frac{1}{x^2+1}\,dx$ converges.

By Weierstrass M-test, $\displaystyle\int_0^{+\infty} \frac{1}{(x^2+\alpha)^2}\,dx$ conv. uniformly.

Since $\displaystyle\int_0^{+\infty} \frac{1}{x^2+\alpha}\,dx = \frac{\pi}{2\sqrt{\alpha}}$

$$\frac{d}{d\alpha}\int_0^{+\infty} \frac{1}{x^2+\alpha}\,dx = \frac{d}{d\alpha}\left(\frac{\pi}{2\sqrt{\alpha}}\right)$$

Hence, $$-\frac{\pi}{4}\alpha^{-3/2} \overset{3.16}{=} \int_0^{+\infty} \frac{\partial}{\partial\alpha}\left(\frac{1}{x^2+\alpha}\right)dx$$

$$= -\int_0^{\infty} \frac{1}{(x^2+\alpha)^2}\,dx$$

$$\therefore \int_0^{+\infty} \frac{1}{(x^2+\alpha)^2}\,dx = \frac{\pi}{4}\alpha^{-3/2}$$

By Thm 3.14,

$$\lim_{\alpha\to 1^+}\int_0^{+\infty} \frac{1}{(x^2+\alpha)^2}\,dx = \int_0^{+\infty}\left(\lim_{\alpha\to 1^+} \frac{1}{(x^2+\alpha)^2}\right)dx = \int_0^{+\infty} \frac{1}{(x^2+1)^2}\,dx = \frac{\pi}{4} \ \#$$

Ex Evaluate $\int_0^{\pi/2} \sqrt{1+4\sin^2 x}\, dx$ in the form of elliptic integral.

Sol$^{\underline{n}}$ Let $I = \int_0^{\pi/2} \sqrt{1+4\sin^2 x}\, dx$

$$I = \int_0^{\pi/2} \sqrt{1+4(1-\cos^2 x)}\, dx$$

$$= \int_0^{\pi/2} \sqrt{5-4\cos^2 x}\, dx$$

Set $x = \frac{\pi}{2} - y \Rightarrow \cos\left(\frac{\pi}{2} - y\right) = \sin y$

and $dx = -dy$

$$I = -\int_{\pi/2}^{0} \sqrt{5-4\sin^2 y}\, dy = \int_0^{\pi/2} \sqrt{5-4\sin^2 y}\, dy$$

$$= \sqrt{5}\int_0^{\pi/2} \sqrt{1-\frac{4}{5}\sin^2 y}\, dy$$

$$= \sqrt{5}\, E\left(\frac{2}{\sqrt{5}}\right)$$

From $k = \frac{2}{\sqrt{5}} = \sin\psi \Rightarrow \psi = \arcsin\left(\frac{2}{\sqrt{5}}\right)$

$$\approx 63^\circ$$

$$= \sqrt{5}(1.1826)$$

$$\approx 2.644 \qquad \#$$

Ex Evaluate $\displaystyle\int_1^{+\infty} \frac{1}{\sqrt{(u^2-1)(u^2+3)}}\,du$

Sol^n Let $u = \sec\theta$; $du = \sec\theta\tan\theta\,d\theta$

$u = 1 \to \theta = 0$, $u = +\infty \to \theta = \frac{\pi}{2}$

$$\int_1^{+\infty} \frac{1}{\sqrt{(u^2-1)(u^2+3)}}\,du = \int_0^{\pi/2} \frac{\sec\theta\tan\theta\,d\theta}{\sqrt{(\sec^2\theta-1)(\sec^2\theta+3)}}$$

$$= \int_0^{\pi/2} \frac{\cancel{\sec\theta\tan\theta}\,d\theta}{\cancel{(\tan\theta)}\sqrt{(\sec^2\theta+3)}}$$

$$= \int_0^{\pi/2} \frac{\sec\theta}{\sqrt{\sec^2\theta+3}}\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{1+3\cos^2\theta}}\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{1+3(1-\sin^2\theta)}}\,d\theta$$

$$= \int_0^{\pi/2} \frac{1}{\sqrt{4-3\sin^2\theta}}\,d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \frac{1}{\sqrt{1-\frac{3}{4}\sin^2\theta}}\,d\theta$$

$$= \frac{1}{2}K\left(\frac{\sqrt{3}}{2}\right) \qquad \#$$

$k = \frac{\sqrt{3}}{2} \Rightarrow \sin\psi = \frac{\sqrt{3}}{2} \Rightarrow \psi = 60^\circ$